C*-algebras generated by isometries: 60 years and counting

M. Laca

University of Victoria

2 April 2021

Isometries on Hilbert space

$V : \mathcal{H} \to \mathcal{H} \quad \|Vh\| = \|h\|,$ equivalently $V^*V = 1.$

V is unitary if $VV^* = 1$ and proper if $VV^* \neq 1.$

V is pure if $V^n V^{*n} \searrow 0$ strongly, i.e. $\bigcap_n V^n \mathcal{H} = \{0\}$

Note: Proper isometries can only exist in infinite dimensions!

The bilateral shift $U : \delta_n \mapsto \delta_{n+1}$ on $\ell^2(\mathbb{Z})$ is a unitary isometry.

The unilateral shift $S : \delta_n \mapsto \delta_{n+1}$ on $\ell^2(\mathbb{N})$ is a pure isometry.

Wold decomposition: Every isometry V on a Hilbert space is unitarily equivalent to a direct sum:

$$\begin{aligned} V &\cong (V_{proper}) \oplus V_{unitary} \\ &\cong (\text{multiple of } S) \oplus (\text{unitary}) \end{aligned}$$

Semigroups of isometries

The collection of all isometries on a Hilbert space (or in a C*-algebra) is a unital left cancellative semigroup:

unital: $I^* I = I$

left-cancellative: $V^*(VW) = W$

semigroup: $(VW)^* VW = W^* V^* VW = 1$

If we want to consider the C*-algebra generated by a collection of isometries, we may as well look at the C*-algebra generated by the left cancellative semigroup determined by that collection.

In other words, study C*-algebras generated by isometric representations of left cancellative semigroups.

Example: the left regular representation

We will assume $e \in P \subset G$ (unital subsemigroup of a group).

The *left regular representation* of P on $\ell^2(P)$ is a representation $p \mapsto L_p$ by isometries given by

$$L_p \delta_q = \delta_{pq} \qquad p \in P$$

on the standard o.n. basis $\{\delta_q\}$ of $\ell^2(P)$ (extended by linearity and continuity).

The C*-algebra generated by the "left creation operators" L_p is called the *reduced Toeplitz C*-algebra* and is our object of interest:

$$\mathcal{T}_\lambda(P) := C^*(L_p : p \in P)$$

$\mathcal{T}_\lambda(P) \subset \mathcal{B}(\ell^2(P))$ so we may use spatial techniques.

Problem:

It is difficult to estimate the norm of an operator, so it is also difficult to decide whether given isometries give rise to a representation of $\mathcal{T}_\lambda(P)$:   if V_k are 'other isometries' representing P, then we would like to know when

$$\|p(V_1, \cdots, V_k; V_1^*, \cdots, V_k^*)\| \overset{?}{\leq} \|p(L_1, \cdots, L_k; L_1^*, \cdots, L_k^*)\|$$

$\mathcal{T}_\lambda(P)$ often satisfies a uniqueness theorem, so in practice this can be estimated sometimes.

Let's look at three classical results, then come back and see a possible strategy to get around this difficulty.

Coburn's Theorem

Theorem (cf. Coburn '67)

Let S be the unilateral shift and suppose V is an isometry. Then the map

$$S \mapsto V$$

extends to a C-algebra homomorphism*

$$\pi_V : C^*(S) \to C^*(V)$$

and π_V is an isomorphism iff V is proper (i.e. $1 - VV^* \neq 0$).

For each V, the map $n \mapsto V^n$ is an isometric representation of $\mathbb{N}$.

Restated: The C*-algebra $\mathcal{T}_\lambda(\mathbb{N}) = C^*(S)$ generated by S has the universal property for isometric representations of $\mathbb{N}$ and the C*-algebra generated by a proper isometry is canonically unique.

Key ingredients in the proof

Lemma: Suppose W is a unitary and S is the unilateral shift.
If $A \oplus B \in C^*(S \oplus W)$, then $\|B\| \leq \|A\|$.

Main inequality: for each polynomial $p(x, y)$ on two noncommuting variables:

$$\|p(W, W^*)\| \leq \|p(S, S^*)\|$$

The rest of the proof consists of applying the Wold decomposition and spatial techniques to a generic isometry $V = S_\mu \oplus V_{unitary}$.

Douglas' Theorem

Theorem [Douglas '72]: Let $\Gamma \subset \mathbb{R}$ be a group, $\Gamma^+ := \Gamma \cap [0, \infty)$,
$L : \Gamma^+ \to \mathcal{B}(\ell^2(\Gamma^+))$ $=$ left regular representation.
Suppose V is an isometric representation of Γ^+. Then the map

$$L_p \mapsto V_p$$

extends to a C*-algebra homomorphism

$$\mathcal{T}_\lambda(\Gamma^+) \xrightarrow{\pi_V} C^*(V)$$

and π_V is an isomorphism iff V is proper
(i.e. iff $1 - V_p V_p^* \neq 0$ for some and hence all $p \in \Gamma^+ \setminus \{0\}$).

Interpretation of Douglas' proof

$B_L := \; C^*(L_pL_p^* : p \in \Gamma^+) \; \cong \; \overline{\mathrm{span}}\{1_{[p,\infty)} \in \ell^\infty(\Gamma^+) : p \in \Gamma^+\}$
there is a faithful conditional expectation (the 'diagonal' map)

$$E_L : \mathcal{T}_\lambda(P) \to B_L$$

If V is a representation of Γ^+ by nonunitary isometries, then

$$\begin{array}{ccc} \mathcal{T}_\lambda(\Gamma^+) & \xrightarrow{\pi} & C^*(V) \\ \Big\downarrow E_L & & \Big\downarrow E_V \\ B_L & \xrightarrow{\pi|_{B_L}} & B_V \end{array}$$

is a commuting square in which the bottom horizontal arrow is an isomorphism.

Cuntz's Theorem

Theorem [Cuntz '81] The C*-algebra generated by n isometries $S_1, S_2, \ldots, S_n$ such that $\sum_j S_j S_j^* < 1$ is
(1) universal for C*-algebras generated by n isometries with mutually orthogonal ranges and
(2) canonically unique.

Note: This is not Cuntz's celebrated theorem about the $\mathcal{O}_n$ from '77. (but is a consequence of it that is needed in computation of K theory using Toeplitz-like extensions).

Since

$$n \text{ isometries} \longleftrightarrow \text{isometric representation of } \mathbb{F}_n^+$$

we can further align this theorem to Coburn's and Douglas' results.

Cuntz's Theorem restated

Theorem

Let $L : \mathbb{F}_n^+ \to \mathcal{B}(\ell^2(\mathbb{F}_n^+))$ be the left regular representation of the free monoid on n generators $\{1, 2, \ldots n\}$. Suppose V is an isometric representation of $\mathbb{F}_n^+$ such that $\sum_j V_j V_j^* \leq 1$ (equivalently the generating isometries have mutually orthogonal ranges).

Then the map

$$L_p \mapsto V_p$$

extends to a C*-algebra homomorphism

$$\mathcal{T}_\lambda(\mathbb{F}_n^+) \xrightarrow{\pi_V} C^*(V)$$

and π_V is an isomorphism if and only if $\prod_j (1 - V_j V_j^*) \neq 0$.

Note the two 'discrepancies': the theorem does not apply to *every* isometric representation, and the characterization of isomorphism does not just say that each isometry has to be proper.

They have to be *jointly proper*.

Cuntz's proof

Sketch: Suppose $\sum_j V_j V_j^* < 1$; amplify $\operatorname{ran}(1 - \sum_j V_j V_j^*)$ to an infinite dimensional subspace $\mathcal{H}_{n+1}$ to make room for an extra isometry $V_{n+1} : \mathcal{H} \to \mathcal{H}_{n+1}$, so that $\sum_{j=1}^{n+1} V_j V_j = 1$, and then use uniqueness of $\mathcal{O}_{n+1}$. $\square$

Of course this only makes us wish we could recall the proof of uniqueness of $\mathcal{O}_n$...

The argument in that proof is similar to the one we saw in the "interpretation of Douglas' proof" and involves the realization of $\mathcal{O}_n$ as a semigroup crossed product $\mathrm{UHF}(n^\infty) \rtimes \mathbb{N}$ (realized as a corner in a crossed product by an action of $\mathbb{Z}$).

Recap and fast-forward to the 90's

We have seen three similar theorems with similar proofs (strictly speaking, Coburn's proof is not similar, but such a proof is possible because the result is a particular case of Douglas'.)
But there was something a bit different for $\mathbb{F}_n^+$
(there was a restriction on the class of representations and a modification of the properness requirement.)

Nica resolved this in '92, by creating a single context for these and many other new results (which also explains the "discrepancies");

The key idea is that of a quasi-lattice ordered group:

Definition: The group/subsemigroup pair (G, P) is a quasi lattice ordered group if $P^2 \subset P$; $P \cap P^{-1} = \{e\}$ and for every $p, q \in G$

$$pP \cap qP = \begin{cases} zP & \text{for some } \; z \in G \qquad (z = p \vee q) \\ \emptyset \end{cases}$$

Nica-covariant representations

The left regular representation $L : P \to \mathcal{B}(\ell^2(P))$ of a QLO semigroup P satisfies

$$L_p L_p^* = \mathbb{1}_{pP}$$

seen as a multiplication operator on $\ell^2(P)$. Hence for $p, q \in P$,

$$(L_p L_p^*)(L_q L_q^*) = \begin{cases} L_z L_z^* & \text{when } zP = pP \cap qP (\neq \emptyset) \\ 0 & \end{cases} \tag{1}$$

Nica's insight was to *require* this property of representations:

Definition [Nica '92] An isometric representation V of P is *covariant* if (1) holds.

Nica covariance in some examples

Let V be an isometric representation of the semigroup P.

(1) If $P =$ total order (e.g. $\mathbb{N}$ and Γ^+), then V is automatically Nica covariant.

(2) If $P = \mathbb{F}_n^+$, then V is Nica covariant iff the generators have mutually orthogonal ranges, in which case the isomorphism condition $1 - \sum_{j=1}^n V_j V_j^* \neq 0$ becomes $\prod_{j=1}^n (1 - V_j V_j^*) \neq 0$ which is a *joint properness* condition.

(3) If $P = \mathbb{N}^d$, then V is Nica covariant iff the generators commute with each other's adjoints; the corresponding isomorphism condition is also $\prod_{j=1}^n (1 - V_j V_j^*) \neq 0$ [H. Salas '85]

An amenability condition for QLO

Let (G, P) be a QLO and consider the universal C*-algebra $C^*_u(G, P)$ for Nica-covariant isometric representations of P: it is generated by a Nica covariant representation v of P and whenever V is another Nica covariant representation, the map $v_p \mapsto V_p$ extends to $C^*_u(G.P) \to C^*(V)$.

There is a conditional expectation

$$E^G : C^*_u(G, P) \longrightarrow C^*(\mathbb{1}_{pP} \in \ell^\infty(P) : p \in P)$$

If E^G is faithful as a positive map, we say that (G, P) is Nica amenable (or satisfies weak containment).

Significantly, this property can be verified directly (and often easily), e.g. if G is abelian. Because then E^G is a Haar measure average over the compact abelian group $\hat{G}$.

Universal property and uniqueness for QLO

Theorem (Nica '92, cf. L-Raeburn '96)

Let L be the left regular representation of a Nica amenable QLO (G, P), and suppose V is a Nica covariant representation of P. Then the map

$$L_p \mapsto V_p \qquad (p \in P)$$

extends to a C-algebra homomorphism*

$$\mathcal{T}_\lambda(P) \xrightarrow{\pi_V} C^*(V)$$

and π_V is an isomorphism iff

$$\prod_{p\in F}(1 - V_p V_p^*) \neq 0$$

for every finite subset $F \subset P \setminus \{e\}$ (i.e. iff V is jointly proper).

(If P is finitely generated, it is enough to take $F = \{\text{generators}\}$.)

Sketch of the proof

Let $C^*_u(G,P)$ be the universal C*-algebra for Nica covariant representations of P. So for V Nica covariant, $v_p \mapsto V_p$ extends to a C*-algebra homomorphism $\pi_V : C^*_u(G,P) \to C^*(V)$.

If, in addition V is *jointly proper*, then there is a commuting diagram with vertical conditional expectation ϕ and faithful bottom horizontal arrow $\pi_V|_D$:

$$\begin{array}{ccc} C^*_u(G,P) & \xrightarrow{\pi_V} & C^*(V) \\ \Big\downarrow E^G & & \Big\downarrow \phi \\ D := C^*(v_p v_p^* : p \in P) & \xrightarrow{\pi_V|_D} & C^*(V_p V_p^* : p \in P). \end{array}$$

Now the familiar argument works:

$\pi_V(b) = 0 \implies \phi \circ \pi_V(b^* b) = 0 \implies (\pi_V|_D) \circ E^G(b^* b) = 0,$
which, for Nica-amenable (G,P), implies $b^* b = 0$ and thus $b = 0$.

Now take $V = L$ to conclude that $\mathcal{T}_\lambda(P) \cong C^*_u(G,P)$ is universal.

Nica amenability: examples and non examples

Examples:

Every QLO (G, P) with G amenable.

$(\mathbb{F}_n, \mathbb{F}_n^+)$.

Free products of QLO with G amenable.

Right-angled Artin monoids.

Graph products of QLO with G amenable.

... and many other examples... but

Non-examples:

Finite type Artin monoids are not .

“Intermediate types" of Artin monoids are not.

... there are many other non examples.

Xin Li's constructible right ideals

By the late 00's many monoids that are not QLO eventually came into focus, especially those arising as $ax + b$ monoids of algebraic integers in fields of class number > 1, e.g $\mathbb{Z}(\sqrt{-5})$

Their Toeplitz C*-algebras were studied directly by 'ad hoc' methods [cf. Cuntz-Deninger-L. '13].

But it soon became evident that for a general theory one would have to generalize Nica-covariance, for which a new idea was needed.

Xin Li provided the new framework by introducing the constructible right ideals associated to a semigroup and studying various C*-algebras of representations that were compatible with their structure.

Constructible right ideals: motivation by example

What is $L_p^* L_q L_r^* L_s$ as an operator on $\ell^2(P)$ if $p^{-1}qr^{-1}s = e$?

$(L_p^* L_q L_r^* L_s)\delta_x = L_p^* L_q L_r^* \delta_{sx}$, and is 0 unless $sx \in rP$, in which case

$L_p^* L_q L_r^* \delta_{sx} = L_p^* \delta_{qr^{-1}sx}$, and is 0 unless $qr^{-1}sx \in pP$, in which case

$L_p^* L_q L_r^* L_s \delta_x = \delta_{p^{-1}qr^{-1}sx} = \delta_x$, because $p^{-1}qr^{-1}s = e$.

Thus,

$$L_p^* L_q L_r^* L_s \delta_x = \begin{cases} \delta_x & \text{if } x \in P \cap s^{-1}rP \cap s^{-1}rq^{-1}pP, \\ 0 & \text{otherwise.} \end{cases}$$

In other words,

$$L_p^* L_q L_r^* L_s = \mathbb{1}_{K(p,q,r,s)},$$

the operator of multiplication by the characteristic function of the set $K(p, q, r, s) := P \cap s^{-1}rP \cap s^{-1}rq^{-1}pP$.

Xin Li's $C_s^*(P)$

Definition: A constructible right ideal in P is a set of the form

$$K(\alpha) := P \cap (p_{2k}^{-1} p_{2k-1})P \cap (p_{2k}^{-1} p_{2k-1} p_{2k-2}^{-1} p_{2k-3})P \cap \cdots \cap (\dot{\tilde{\alpha}})P.$$

with $p_1, \ldots, p_2 k \in P$ and $\dot{\tilde{\alpha}} := p_{2k}^{-1} p_{2k-1} \cdots p_2^{-1} p_1$

The set of constructible right ideals is a semi-lattice under intersections.

Definition [X. Li '12] Let $C_s^*(P)$ be the universal C*-algebra generated by an isometric representation $\{v_p : p \in P\}$ and projections e_S for each constructible right ideal S, such that for instance when $p^{-1} q r^{-1} s = e$, then the product $v_p^* v_q v_r^* v_s$ is e_S for $S = P \cap s^{-1} r P \cap s^{-1} r q^{-1} p P$.

A uniqueness result of Xin Li's

The C*-algebra with this presentation has many nice features, and is isomorphic to $\mathcal{T}_\lambda(P)$ in several interesting examples, notably those arising from $ax + b$-monoids of algebraic integers. Here is one of Li's theorems, cast in a familiar form.

Theorem [Li '17] Suppose P is a submonoid of a group G such that $P \cap P^{-1} = \{e\}$. Let $(L_p, \mathbb{1}_S)$ be the left regular representation of P, and suppose (V_p, E_S) is a representation of P that is 'covariant in the sense of Li for constructible right ideals' and suppose there is a representation

$$\pi_{V,E} : \mathcal{T}_\lambda(P) \to C^*(V, E)$$

$$L_p \mapsto V_p, \qquad \mathbb{1}_S \mapsto E_S$$

Then $\pi_{V,E}$ is faithful if and only if its restriction to the diagonal $D_\lambda = C^*(\mathbb{1}_S : S = \text{constructible r-ideal})$ is faithful.

Further work

However... in addition to the unavoidable issue of amenability, the theorem does not admit any invertibles in P.

Moreover, Li's construction $C_s^*(P)$ is fully satisfactory as universal C*-algebra for a class of representations only when P satisfies a condition called independence which boils down to linear independence of the characteristic functions of constructible right ideals.

This leads to the joint work with Sehnem... but this will have to wait until the second talk.

Thanks!