# Fourier algebra of $\mathbb{R} \rtimes \mathbb{R}^{\times}$ and a dual convolution


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## Non-commutative harmonic analysis

Function spaces related to Fourier analysis of non-abelian groups. E.g.

- Fourier alg, Rajchman alg, weighted Fourier alg.
- Study of their derivations, spectrum, and amenability.

## My plan:

- What is the Fourier algebra? What do we know about it?
- Describe the Fourier algebra of $\mathbb{R} \rtimes \mathbb{R}^{\times}$, and define a dual convolution for it.
- Construct symmetric derivation on Fourier algebra of $\mathbb{R} \rtimes \mathbb{R}^{\times}$.

# Classical harmonic analysis

Let $G$ be abelian, and

$$\widehat{G} = \Big\{ \chi : G \to \mathbb{T} : \text{ continuous } \ \& \ \chi(xy) = \chi(x)\chi(y) \Big\}.$$

## Fourier transform

- $\mathcal{F} : L^1(G) \to C_0(\widehat{G})$ given by $\widehat{f}(\xi) = \int_G f(x)\overline{\xi(x)}dx$ is a cts *-hom.
- $\mathcal{F}(f * g) = \widehat{f} \ \ \widehat{g}$.

## Fourier algebra

$\mathrm{A}(\widehat{G}) = \mathcal{F}(L^1(G)), \quad \|\widehat{f}\|_{\mathrm{A}} = \|f\|_1.$

# Coefficient function spaces

$G$: locally compact group. E.g. $\mathbb{Z}^n$, $\mathbb{R}^n$, $\mathrm{SU}(2)$, $\mathrm{SL}(2,\mathbb{R})$, ...

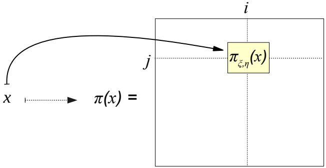
- A continuous unitary representation of $G$ on a Hilbert space $\mathcal{H}$ is

$$\boxed{\pi : G \to \mathcal{U}(\mathcal{H}),\ \pi(xy) = \pi(x)\pi(y).}$$

- $\widehat{G} = \Big\{\text{equivalence classes of irreducible } \pi\Big\}.$
- A coefficient function associated with $\pi$ is

$$\boxed{\pi_{\xi,\eta} : G \to \mathbb{C},\ \pi_{\xi,\eta}(x) := \langle \pi(x)\xi, \eta\rangle.}$$

**Ex.** Let $\mathcal{H} = \mathbb{C}^n$ with standard basis $\{e_i\}_{i=1}^n$, $\xi = e_i$ and $\eta = e_j$.



# Coefficient function spaces

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**Example**

Regular representation $\lambda : G \to \mathcal{U}(L^2(G))$ is defined as

$$(\lambda(x)f)(y) = f(x^{-1}y).$$

**Peter-Weyl Theorem for compact groups**

Regular rep decomposes into irreducible reps.

# Fourier algebra [Eymard, 1964]

**Recall.** $\lambda_{f,g}(x) = \langle \lambda(x)f, g\rangle_{L^2(G)}$.

**Fourier algebra**

$$\mathrm{A}(G) := \left\{\lambda_{f,g} : \ f, g \in L^2(G)\right\} \subseteq C_0(G).$$

- This is a commutative Banach algebra (pointwise product).
- $\|u\|_{\mathrm{A}(G)} = \inf \left\{\|f\|_2\|g\|_2 : u = \lambda_{f,g}\right\}$.
- $\mathrm{A}(G)^* \simeq \mathrm{VN}(G)$.

- If $G$ is abelian, then
$$\mathrm{A}(G) = \left\{f \in C_0(G) : \ f = \widehat{h} \ \text{ for some } \ h \in L^1(\widehat{G})\right\}.$$
- If $G$ compact, then
$$\mathrm{A}(G) = \left\{f \in C(G) : \sum_{\pi\in\widehat{G}} d_\pi \|\pi(f)\|_1 < \infty\right\}.$$

$\pi(f) = \int_G f(x)\pi(x)dx$.

The Banach algebra $\mathrm{A}(G)$ encodes $G$.

- [Eymard, 1964] $\sigma(\mathrm{A}(G)) \simeq G$ (set-wise and topologically).
- [Walter, 1970] $\mathrm{A}(G_1) \simeq \mathrm{A}(G_2)$ iff $G_1 \simeq G_2$.
- [Leptin, 1968] $\mathrm{A}(G)$ has b.a.i. iff $G$ amenable.
  (e.g. $G$ compact or abelian or solvable...)

**My focus**

Existence or non-existence of "derivations" on $\mathrm{A}(G)$ has many implications in $G$.

**Theorem [Connes and Haagerup]**

For $C^*$-algebras, "lack of derivations" and nuclearity coincide.

## Definition

$D : \mathrm{A}(G) \to X$ is derivation if linear, continuous, and

$$D(uv) = D(u) \cdot v + u \cdot D(v)$$

**Note.** $X$ is a Banach $\mathrm{A}(G)$-bimodule, *i.e.* $\|u \cdot x\|, \|x \cdot u\| \leq C\|x\| \, \|u\|$.

## When do derivations exist?

- $\nexists$ derivation $D : \mathrm{A}(G) \to \mathbb{C}$, as it would be $f \mapsto f'(x_0)$.
- $\nexists$ derivation $D : \mathrm{A}(G) \to \mathrm{A}(G)$. [Singer-Wermer, 55]

## Open question

For which $G$ $\exists$ nonzero derivation $D : \mathrm{A}(G) \to \mathrm{A}(G)^*$?

**Open question**

For which $G$ $\exists$ nonzero derivation $D : \mathrm{A}(G) \to \mathrm{A}(G)^*$?

**Note.** $\mathrm{A}(G)^*$ is an important *symmetric* $\mathrm{A}(G)$-bimodule.

Question was reduced to connected groups.
[Forrest 88, Johnson 94, Forrest-Runde 04]

**Conjecture, 1995**

$G$ connected group. Then such $D$ exists iff $G$ non-abelian.

Conjecture confirmed for compact connected groups, by reducing to Lie cases. [**Johnson, 94**; Forrest-Samei-Spronk, 09]

# Non-compact groups; semisimple Lie

**Theorem [Choi-G., 2014]**

If $G$ non-compact, connected semisimple Lie, then $\mathrm{A}(G)$ has a nonzero derivation.

$G$: group of affine transformations on the real line (connected).

$$\mathbb{R} \rtimes \mathbb{R}_+^\times = \left\{ \begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix} : \ a \in \mathbb{R}^{>0}, b \in \mathbb{R} \right\}$$

**Theorem [Choi-G., 2014]**

We find the first explicit $\mathrm{A}(\mathbb{R} \rtimes \mathbb{R}_+^\times) \to \mathrm{A}(\mathbb{R} \rtimes \mathbb{R}_+^\times)^*$, for non-cpt $G$.

$$D(f) : \mathrm{A}(\mathbb{R} \rtimes \mathbb{R}_+^\times) \to \mathrm{A}(\mathbb{R} \rtimes \mathbb{R}_+^\times)^*,$$
$$D(f)(g) = \int_{\mathbb{R} \rtimes \mathbb{R}^\times} (\text{suitable function}) \frac{\partial f}{\partial b}(b,a) g(b,a)\, d(b,a).$$

**Note.** $\mathrm{A}(\mathbb{R} \rtimes \mathbb{R}^\times) \to \mathbb{C}, \ f \mapsto \frac{\partial f}{\partial b}$ is not a bounded derivation.

# Problem solved!

**Theorem [Lee-Ludwig-Samei-Spronk, 2016]**

If $G$ connected non-abelian Lie group, then $\mathrm{A}(G)$ has nonzero derivations.

**Idea of proof**

By structure theory of connected Lie groups, the problem reduces to

1. Affine group of $\mathbb{R}$,
2. Heisenberg group,
3. Euclidean motion group,
4. Grélaud group.

**Remark**

The condition “Lie” was dropped by Victor Losert (2020?).

# Understanding space of derivations

Candidates $A \rtimes H$, where $H \leq \mathrm{GL}_n(\mathbb{R})$.

**Typical example: group of affine translations on $\mathbb{R}$**

$$G = \mathbb{R} \rtimes \mathbb{R}^\times = \left\{ \begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix} : \ a \in \mathbb{R}^\times, b \in \mathbb{R} \right\}.$$

- $\mathbb{R}^\times := (\mathbb{R}_{\neq 0}, *)$ with Haar measure $\frac{dt}{|t|}$. $\mathbb{R}^\times$ acts on $\mathbb{R}$ by mult.
- $G \simeq \mathbb{R} \rtimes \mathbb{R}^\times = \{(b, a) : b \in \mathbb{R}, a \in \mathbb{R}^\times\}$.

**$\infty$-dim irred rep**

Mackey machine for induced rep $\pi : G \to \mathcal{U}(L^2(\mathbb{R}^\times, dt/|t|))$,

$$\pi(b, a)\xi(t) = e^{2\pi ibt}\xi(at).$$

- $\lambda \simeq \infty \cdot \pi$.
- coefficient functions $\pi_{\xi,\eta}(b, a) = \int_0^\infty e^{2\pi ibt}\xi(at)\overline{\eta(t)}\,\frac{dt}{|t|}$.
- $\pi$ is square integrable, *i.e.* $\pi_{\xi,\eta} \in \mathrm{A}(G)$.

# Fourier analysis of $\mathbb{R} \rtimes \mathbb{R}^\times$

**Recall:** $\pi : G \to \mathcal{U}(L^2(\mathbb{R}^\times, dt/|t|))$, $\pi(b,a)\xi(t) = e^{2\pi i b t}\xi(at)$.

## Fourier transform

Let $\mathcal{H} = L^2(\mathbb{R}^\times, dt/|t|)$, and consider the trace-class operators $\mathcal{H}\widehat{\otimes}\overline{\mathcal{H}}$.

$$\text{Fourier transform}\quad \Psi : \mathcal{H}\widehat{\otimes}\overline{\mathcal{H}} \to \mathrm{A}(G), \quad \xi \otimes \eta \mapsto \pi_{\xi,\eta}$$

(Banach space) isometric isomorphism.

## Theorem [Choi-G., 2020]

Define the dual convolution of two rank-one tensors in $\mathcal{H}\widehat{\otimes}\overline{\mathcal{H}}$ as

$$(\xi \otimes \eta) \boxtimes (\xi' \otimes \eta') :=$$

$$\int_{\mathbb{R}^\times} \left(\lambda(1+h)\xi \cdot \lambda(1+h^{-1})\xi'\right) \otimes \left(\lambda(1+h)\eta \cdot \lambda(1+h^{-1})\eta'\right) \frac{dh}{|h|}.$$

Then, $(\mathcal{H}\widehat{\otimes}\overline{\mathcal{H}}, \boxtimes)$ is a Banach algebra.

$$(\xi \otimes \eta) \boxtimes (\xi' \otimes \eta') :=$$

$$\int_{\mathbb{R}^\times} \big(\lambda(1+h)\xi \cdot \lambda(1+h^{-1})\xi'\big) \otimes \big(\lambda(1+h)\eta \cdot \lambda(1+h^{-1})\eta'\big) \ \frac{dh}{|h|} \ .$$

**Theorem [Choi-G., 2020]**

Let $T_1, T_2 \in \mathcal{S}_1(\mathcal{H})$. Then for a.e. $(s,t) \in \mathbb{R}^\times \times \mathbb{R}^\times$

$$(T_1 \boxtimes T_2)(s,t) = \int_{-\infty}^{\infty} T_1\left(\frac{s}{1+h}, \frac{t}{1+h}\right) T_2\left(\frac{s}{1+h^{-1}}, \frac{t}{1+h^{-1}}\right) \frac{dh}{|h|}$$

where the integral is absolutely convergent for a.e. $(s,t) \in \mathbb{R}^\times \times \mathbb{R}^\times$.

**Theorem [Choi-G., 2020]**

Fourier transform $\Psi : \mathcal{H}\widehat{\otimes}\overline{\mathcal{H}} \to \mathrm{A}(G), \quad \xi \otimes \eta \mapsto \pi_{\xi,\eta}$ is a (Banach algebra) isometric isomorphism.

# Derivations on $\mathcal{H}\widehat{\otimes}\overline{\mathcal{H}}$

Define a multilinear map $\Phi : \mathcal{H}\widehat{\otimes}\overline{\mathcal{H}} \otimes \mathcal{H}\widehat{\otimes}\overline{\mathcal{H}} \to \mathbb{C}$ by

$$\Phi(T_1 \otimes T_0) = \int_{\mathbb{R}^\times \times \mathbb{R}^\times} \mathrm{sign}(s) T_1(s,t) T_0(-s,-t) \frac{d(s,t)}{|s||t|}.$$

- View $\Phi$ as a bilinear form on $\mathcal{H}\widehat{\otimes}\overline{\mathcal{H}}$.
- Define $D$ to be the corresponding operator $\mathcal{H}\widehat{\otimes}\overline{\mathcal{H}} \to (\mathcal{H}\widehat{\otimes}\overline{\mathcal{H}})^*$ as

$$D(T_1)(T_0) = \Phi(T_1 \otimes T_0).$$

**Theorem [Choi-G., 2020]**

We have

$$D(T_1 \boxtimes T_2)(T_0) = D(T_1)(T_2 \boxtimes T_0) + D(T_2)(T_1 \boxtimes T_0).$$

So $D$ is a derivation on the Banach algebra $\mathcal{H}\widehat{\otimes}\overline{\mathcal{H}}$.

# Derivations on $\mathrm{A}(\mathbb{R} \rtimes \mathbb{R}^\times)$

- Fourier transform: $\Psi : \mathcal{H}\widehat{\otimes}\overline{\mathcal{H}} \to \mathrm{A}(G), \quad \xi \otimes \eta \mapsto \pi_{\xi,\eta}$.
- Derivative: $D : \mathcal{H}\widehat{\otimes}\overline{\mathcal{H}} \to (\mathcal{H}\widehat{\otimes}\overline{\mathcal{H}})^*$,

$$D(T_1)(T_0) = \int_{\mathbb{R}^\times \times \mathbb{R}^\times} \mathrm{sign}(s) T_1(s,t) T_0(-s,-t) \frac{d(s,t)}{|s||t|}.$$

**Theorem [Choi-G., 2020]**

$\widetilde{D} := (\Psi^{-1})^* \circ D \circ \Psi^{-1}$ is a derivation on the Ban algebra $\mathrm{A}(\mathbb{R} \rtimes \mathbb{R}^\times)$.

Thank you very much!