

Recall

1) X, A finite sets,

$$\mathcal{S}_{X,A} = \text{span}\{e_{x,a} : x \in X, a \in A\} \subseteq \bigoplus_{x \in X} \mathcal{C}_A =: \mathcal{L}_{X,A}$$

$$\sum_a e_{x,a} = 1, \quad \forall x$$

Captures families of POVM's:

$$\phi: \mathcal{S}_{X,A} \xrightarrow{\text{map}} \mathcal{B}(H) \iff (E_{x,a})_{\substack{a \in A \\ x \in X}} \text{POVM}$$

2) $\mathcal{S}_{X,A} \otimes_C \mathcal{S}_{Y,B} \subseteq \mathcal{L}_{X,A} \otimes_{\max} \mathcal{A}_{Y,B}$
as an operator subsystem.

3) $\mathcal{S}_{X,A} \otimes_{\min} \mathcal{S}_{Y,B} \subseteq \mathcal{L}_{X,A} \otimes_{\min} \mathcal{A}_{Y,B}$
as an operator subsystem.

4) NS correlations, \mathcal{L}_{NS}

$$\mathcal{P} = \left\{ (\rho(a,b|x,y))_{a,b} : (x,y) \in X \times Y \right\}$$

$$\text{s.t. } \sum_b \rho(a,b|x,y) =: \rho(a|x), \quad \forall y \in Y$$

$$\sum_a \rho(a,b|x,y) =: \rho(b|y), \quad \forall x \in X.$$

5) $\mathcal{L}_{QC} : \rho(a,b|x,y) = \langle E_{x,a} F_{y,b} \rangle_{\text{S-S}}$
 $(E_{x,a})_{a \in A}, \quad (F_{y,b})_{b \in B}$ commuting POVM's.

$$6) \quad \mathcal{C}_2 : \quad \langle (E_{x,a} \otimes F_{y,b})_{\xi, \zeta} \rangle$$

For $s : S_{X,A} \otimes S_{Y,B} \rightarrow \mathbb{C}$, define

$$\varphi_s(a, b | x, y) := s(E_{x,a} \otimes F_{y,b})$$

$f_{j,b}$: generator of $S_{Y,B}$.

Theorem $s \rightarrow \varphi_s$ bijective affine correspondence

$$(i) (S_{X,A} \otimes_{\max} S_{Y,B})^d \rightarrow \mathcal{C}_{ns};$$

$$(ii) (S_{X,A} \otimes_{\sim} S_{Y,B})^d \rightarrow \mathcal{C}_{\Sigma^c};$$

$$(iii) (S_{X,A} \otimes_{\min} S_{Y,B})^d \rightarrow \mathcal{C}_{\Sigma^a}.$$

Def Assume $Y = X$ and $B = A$.

$\varphi \in \mathcal{C}_{ns}$ synchronous if

$$\varphi(a, b | x, x) = 0 \text{ if } a \neq b.$$

Theorem Let $\varphi = \{(q(a, b | x, y))_{a, b \in A : x, y \in X}\}$

synchronous. Then \exists trace state

$\pi : A_{X,A} \rightarrow \mathbb{C}$ such that

$$\varphi(a, b | x, y) = \pi(E_{xa} E_{yb}), \quad x, y \in X \\ a, b \in A.$$

Theorem $\hookrightarrow \varphi$ is bijective affine correspondence

$$(i) (\mathcal{S}_{X,A} \oplus_{\max} \mathcal{S}_{Y,B})^d \rightarrow \ell_{ns};$$

$$(ii) (\mathcal{S}_{X,A} \oplus_{\sim} \mathcal{S}_{Y,B})^d \rightarrow \ell_{\Sigma c};$$

$$(iii) (\mathcal{S}_{X,A} \oplus_{\min} \mathcal{S}_{Y,B})^d \rightarrow \ell_{\Sigma a}.$$

PF (i) $\ell_{ns} \ni p$.

$$p(a|x) \quad \sum_a p(a|x) = 1 \quad \forall x.$$

$$\Rightarrow (\lambda_{x,a}) \subseteq \mathbb{C} \text{ sat.}$$

$$\sum_a \lambda_{x,a} = \sum_a \lambda'_{x,a} \quad \forall x, x'$$

$$\mathcal{S}_{X,A}^d \quad \mathcal{S}_{X,A} = \frac{\ell_{X,A}^\infty}{\ell_A^\infty \oplus \dots \oplus \ell_A^\infty}.$$

$$\mathcal{S}_{X,A}^d \quad \mathcal{S}_{X,A} = \frac{\ell_{X,A}^\infty}{\ell_A^\infty \oplus \dots \oplus \ell_A^\infty}.$$

$$\{(\lambda_{x,a})_{x,a} : \sum_a \lambda_{x,a} = c \quad \forall x\}.$$

$$\mathcal{S}_{X,A}^d \oplus_{\min} \mathcal{S}_{Y,B}^d = \{(\mu_{x,y,c,d}) :$$

have the NS property\}.

$\Rightarrow (S_{X,A} \otimes_{\max} S_{Y,B})^d \ni s \text{ state}$
||

$$S_{X,A}^{\perp} \otimes_{\min} S_{Y,B}^d$$

(II) s state on $S_{X,A} \otimes_c S_{Y,B}$

miss state on $S_{X,A} \otimes_{\max} S_{Y,B}$

miss CNS

$$s(u \otimes v) = \langle \pi(u) p(v) | \xi, \xi \rangle$$

$$\pi(e_{x,a}) =: E_{x,a}, p(f_{j,s}) =: F_{j,s}$$

$$p_s(a, b | x, y) = s(e_{x,a} \otimes f_{y,b}) \\ = \langle E_{x,a} F_{y,b} | \xi, \xi \rangle.$$

Conversely: start with $\varphi \in \ell_{2c}$

$\rightsquigarrow \phi: S_{X,A} \rightarrow B(H)$ map
 $e_{x,a} \rightarrow E_{x,a}$

$\psi: S_{Y,B} \rightarrow B(H)$
 $f_{j,s} \rightarrow F_{j,s}$

$\rightsquigarrow \phi \circ \psi (u \otimes v) := \phi(u) \psi(v) \quad \text{map}$

$$s(u \otimes v) := \langle (\phi \cdot \psi)(u \otimes v), \xi \rangle.$$

$$(fri) \quad s: A_{X,A} \otimes_{min} A_{T,B} \rightarrow C.$$

$$\pi: A_{X,A} \rightarrow B(H)$$

$$\rho: A_{T,B} \rightarrow B(K) \quad \text{faithful.}$$

$$\pi(A_{X,A}) \otimes_{min} \rho(A_{T,B}) \subseteq B(H \otimes K).$$

s can be approximated by convex combinations of vector state. i.e.

$$s(e_{x,a} \otimes f_{j,b}) \approx \sum_{i=1}^m \lambda_i \langle (\tilde{E}_{x,a} \otimes \tilde{F}_{j,b}) \xi_i, \xi_i \rangle.$$

$$K \rightsquigarrow K \otimes \mathbb{C}^m$$

$$\xi = (\sqrt{\lambda_i} \xi_i) . \rightsquigarrow \tilde{H}, \tilde{K}.$$

$$s(e_{x,a} \otimes f_{j,b}) \approx \langle \tilde{E}_{x,a} \otimes \tilde{F}_{j,b} \xi, \xi \rangle.$$

$$\tilde{E}_{x,a}, \tilde{F}_{j,b} : \text{in finite-dim. space}.$$

$$P_n \uparrow I \quad P_n \text{ finite rank}$$

$$Q_n \uparrow I \quad Q_n \rightarrow 0$$

$$\tilde{E}_{x,a}^{(n)} = P_n \tilde{E}_{x,a} P_n$$

$$F_{j,b}^{(n)} = \alpha_n \tilde{F}_{j,b} \alpha_n.$$

$$\langle E_{x,a}^{(n)} \otimes F_{j,b}^{(n)}, \tilde{\gamma}_n \rangle,$$

$$\tilde{\gamma}_n = \frac{\Phi_n \otimes \Phi_n |\xi|}{\|(\Phi_n \otimes \Phi_n) \xi\|},$$

Def Assume $\gamma = X$ and $B = A$.

$\varphi \in \text{Ens}$ synchronous if

$$\varphi(a, b | x, y) = 0 \text{ if } a \neq b.$$

Theorem Let $\varphi \sim \{(q(a, b | x, y))_{a, b \in A : x, y \in X}\}$ be Ens & synchronous. Then \exists trace state $\pi : A_{X,A} \rightarrow \mathbb{C}$ such that

$$\varphi(a, b | x, y) = \pi(E_{x,a} E_{y,b}^*), \quad x, y \in X \\ a, b \in A.$$

$$\mathcal{T}(\underbrace{E_{x,a} E_{y,b}}_y) = 0 \text{ if } a \neq b.$$

Pf Fix φ . Write

$$\varphi(a, b | x, y) = s(E_{x,a} \otimes F_{y,b}), \\ s : A_{X,A} \otimes_{\text{max}} A_{X,A} \rightarrow \mathbb{C}$$

state.

Let $\tau: \mathcal{D}_{X,A} \rightarrow \mathbb{C}$ be

$$\tau(u) := s(u \otimes 1), \quad u \in A_{\times, A}.$$

I claim τ is a trace state.

$$s(e_{x,a} \otimes 1) = \sum_{b \in A} \underbrace{s(e_{x,a} \otimes f_{x,b})}_{0' \neq a \neq b} =$$

$$= s(e_{x,a} \otimes f_{x,a}).$$

$$s(1 \otimes f_{x,a}) = s(e_{x,a} \otimes f_{x,a}).$$

Write $u \sim v$ if $s(u - v) = 0$,

$$u, v \in A_{\times, A} \oplus A_{\times, A}.$$

$$\Rightarrow e_{x,a} \otimes 1 \sim e_{x,a} \otimes f_{x,a} \sim 1 \otimes f_{x,a}.$$

$$h_{x,a} := e_{x,a} \otimes 1 - 1 \otimes f_{x,a}.$$

$$h_{x,a} \sim 0.$$

$$\begin{aligned} h_{x,a}^2 &= e_{x,a} \otimes 1 + 1 \otimes f_{x,a} \\ &\quad - e_{x,a} \otimes f_{x,a} - e_{x,a} \otimes f_{x,a}. \end{aligned}$$

$$\Rightarrow h_{x,a}^2 \sim 0.$$

$$\Rightarrow s(uh_{x,a}) = 0 = s(h_{x,a}w)$$

$\forall u \in A_{x,A} \otimes A_{x,A}$

$z \in A_{x,A}$. [Want: $zw \otimes 1 \sim w z \otimes 1$
 $z, w \in A_{x,A}$]

$$ze_{x,a} \otimes 1 \sim z \otimes f_{x,a} \sim e_{x,a} z \otimes 1.$$

$$(zw)(\underbrace{e_{x,a} \otimes 1 - 1 \otimes f_{x,a}}_{h_{x,a}}) \sim 0.$$

Induction on the length of w

$$w = w'e \quad e \in \{e_{x,a} : x, a\}$$

$$zw \otimes 1 = zw'e \otimes 1 \sim e zw' \otimes 1 \sim w'e z \otimes 1 = wz \otimes 1.$$

$$\begin{aligned}\varphi(a, b | x, y) &= s(e_{xa} \otimes f_{yb}) \\ &= s(e_{xa} e_{yb} \otimes 1) = I(e_{xa} e_{yb})\end{aligned}$$