

Def Op. system: $\mathcal{S} \subseteq \mathcal{B}(H)$, H Hilbert space,

$$T \in \mathcal{S} \Rightarrow T^* \in \mathcal{S},$$

$$1 \in \mathcal{S}.$$

\mathcal{S}, \mathcal{T} op. systems.

$$M_n(\mathcal{S}) \subseteq \mathcal{B}(H^n). \quad M_n(\mathcal{S})^+ = M_n(\mathcal{S}) \cap \mathcal{B}(H^n)^+.$$

$\phi: \mathcal{S} \rightarrow \mathcal{T}$ positive if

$$S \in \mathcal{S}^+ \Rightarrow \phi(S) \in \mathcal{T}^+.$$

ϕ unital if $\phi(1) = 1$.

$\phi^{(n)}: M_n(\mathcal{S}) \rightarrow M_n(\mathcal{T})$ positive $\forall n$.

$$\phi^{(n)}(1_n \otimes S) = (1_n \otimes \phi(S))$$

Thm (Choi - Effros) \forall abstract op. system is an op. system.

Abstract op. sys.

V vector \mathbb{K} -space

$$\forall n: V = V^* \quad (v_j)^* = (v_j^*)$$

$$C_n \in M_n(V)_{\mathbb{R}}, \quad n \in \mathbb{N}$$

$$C_n \cap (-C_n) = \{0\}.$$

$$\alpha \in C_n \text{ and } \alpha^* \in C_m \quad \alpha \in M_{m,n}(\mathbb{C})$$

$$e \in V \text{ A.O.U.} \quad e_n = \begin{matrix} u, u. \\ \begin{bmatrix} e & & \\ 0 & e & \\ & & e \end{bmatrix} \end{matrix}$$

$$u \in M_n(V)_h \Rightarrow u \leq r e_n \text{ for some } r > 0.$$

$$u + r e_n \in C_n \quad \forall r > 0 \Rightarrow u \in C_n.$$

V, W abstract ops.

$$\rightsquigarrow \phi: V \rightarrow W \text{ c.p. (l.c.p.),}$$

Char - Effrs

V abstract \rightsquigarrow system

$$\Rightarrow \exists \phi: V \rightarrow \mathbb{B}(H) \text{ c.p.}$$

$$\exists \phi^{-1}: \phi(V) \rightarrow V \text{ c.p.}$$

States

$$\phi: \mathcal{S} \rightarrow \mathbb{C} \text{ un. positive map}$$

$$x \in \mathcal{S} \quad \|x\| \leq 1 \Leftrightarrow \begin{bmatrix} 1 & x \\ x^* & 1 \end{bmatrix} \succcurlyeq 0.$$

Duals

\mathcal{S} ops

$$\mathcal{S}^d : \phi: \mathcal{S} \rightarrow \mathbb{C} \text{ B.L. linear}$$

$$M_n(\mathcal{S}^d) \ni (f_{ij})$$

$$F: \mathcal{S} \rightarrow M_n$$

$$F(u) = (f_j(u))_{j \cdot}$$

$$(f_{ij}) \in M_n(\mathbb{S}^d)^+ \text{ if } (y \neq \emptyset).$$

F c.p.

Thm (Choi-Effros) If \mathcal{S} is finite dim.
then \mathbb{S}^d is an o.p.s.

Idea Take $\{\omega_k\}$ basis for \mathbb{S}^d , hermitian

$$\rightarrow \omega_k = \omega_k^+ - \omega_k^-$$

ω_k^+, ω_k^- positive f-ls.

$$\text{Take } \omega := \sum \omega_k^+ + \omega_k^- : A \cdot 0 \cdot 0.$$

The quotient

$J \subseteq \mathcal{S}$ **kernel** if \exists c.p. map
 $\phi: \mathcal{S} \rightarrow \mathcal{B}(\mathcal{H})$ s.t. $J \subseteq \ker(\phi)$.

Thm If \mathcal{S} o.p.s., $J \subseteq \mathcal{S}$ kernel

then \exists o.p.s. \mathcal{S}/J s.t.

if $\phi: \mathcal{S} \rightarrow \mathcal{T}$ c.p. $\rightarrow J \subseteq \ker \phi$,
then $\tilde{\phi}: \mathcal{S}/J \rightarrow \mathcal{T}$ c.p.

Idea $(S/J)^+ \ni u+J$.

$\exists v \in J$ s.t. $u+v \in S^+$.

General procedure:

If a cone D does not possess an A.O.V. then

$$\tilde{D} = \{u : u + \epsilon I \in D \ \forall \epsilon > 0\}.$$

\tilde{D} will do.

[Recall positive completion problem:

$$\begin{bmatrix} * & * & ? & * \\ * & * & * & ? \\ * & * & * & * \end{bmatrix} \rightsquigarrow \begin{bmatrix} * & * & - & - \\ - & - & - & - \\ - & - & - & - \end{bmatrix} \text{ positive?}$$

Then If $\phi: S \rightarrow T$ ^{function.} complete isomorphism

$$[\tilde{\phi}: S/J \rightarrow T \text{ c.o.}]$$

then $\phi^d: T^d \rightarrow S^d$ c.o. embedding.

Tensor products

1) Minimal:

$$S \in \mathcal{B}(H), \quad T \in \mathcal{B}(K)$$

$\leadsto \mathcal{S} \otimes_{\min} \mathcal{T} \subseteq \mathcal{R}(H \otimes K)$.

Prop \mathcal{S}, \mathcal{T} f.i. lim.

$u \in (\mathcal{S} \otimes_{\min} \mathcal{T})^+ \Leftrightarrow$

$\exists u: \mathcal{S}^d \rightarrow \mathcal{T} \quad \text{c.p.}$

$$u = \sum_{i=1}^k x_i \otimes f_i$$

$$\leadsto \Theta u(f) = \sum_{i=1}^k f(x_i) f_i$$

2) Maximal \otimes . : smallest cones.

condition: $u \in M_n(\mathcal{S})^+$,
 $v \in M_m(\mathcal{T})^+$

then $u \otimes v \in M_{nm}(\mathcal{S} \otimes \mathcal{T})^+$.

$$D_n = \left\{ \alpha (u \otimes v) \alpha^* : \begin{array}{l} u \in M_k(\mathcal{S})^+ \\ v \in M_\ell(\mathcal{T})^+ \\ \alpha \in M_{n, k\ell} \end{array} \right\}$$

$\leadsto (C_n^{\max})_{n \in \mathbb{N}}$.

Thm If $\phi: \mathcal{S} \times \mathcal{T} \rightarrow \mathcal{B}(H)$
jointly c.o.p. positive then

$$\tilde{\phi} : \mathcal{S} \otimes_{\max} \mathcal{T} \rightarrow \mathcal{B}(\mathcal{H}) \quad \text{is } \mathcal{P}.$$

$$\begin{aligned} & \text{[} \phi^{(nm)}((u_{ij}), (v_{pq})) \in M_{nm}(\mathcal{B}(\mathcal{H}))^+ \\ & \text{if } (u_{ij}) \in M_n(\mathcal{S})^+ \\ & \quad (v_{pq}) \in M_m(\mathcal{T})^+ \\ & \rightarrow (\phi(u_{ij}, v_{pq}))_{ijpq} \end{aligned}$$

Note If A, B C^* -alg. (unital)

$$A \otimes_{\max} B$$

$$A \otimes_{C^* \text{-max}} B.$$

completion
of