Toeplitz algebras for semigroups: universality and uniqueness results

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Today's menu

1. Neutral words and generalized left-quotients on P
2. Constructible right ideals and Xin Li's $C_s^*(P)$
3. Universal Toeplitz algebra $\mathcal{T}_u(P)$
4. Action of P^* on D_λ and faithful representations of $\mathcal{T}_\lambda(P)$
5. Uniqueness theorem for general P
6. Uniqueness for 'topologically free P'
7. (Universal boundary quotient)

Generalized left quotients on a submonoid of a group

Suppose $P \subset G$ is a submonoid of a group and let

$\alpha = (p_1, p_2, \ldots, p_{2k})$ with $p_j \in P$, a word of even length

$\tilde{\alpha} = (p_{2k}, \ldots, p_2, p_1)$ the reverse word

$\dot{\alpha} = p_1^{-1} p_2 p_3^{-1} p_4 \cdots p_{2k-1}^{-1} p_{2k} \in G$ the *generalized left quotient* (or multifraction)

$\dot{W}_\alpha := W_{p_1}^* W_{p_2} W_{p_3}^* \cdots W_{p_{2k-1}}^* W_{p_{2k}}$, $(W_p : p \in P)$ in a C*-algebra

Define

$$K(\alpha) := P \cap p_{2k}^{-1} p_{2k-1} P \cap p_{2k}^{-1} p_{2k-1} p_{2k-2}^{-1} p_{2k-3} P \cap \cdots \cap \dot{\tilde{\alpha}} P$$

Last week:

If $p \mapsto L_p$ denotes the l.r.r. of P and if $\dot{\alpha} = e$, then

$$\dot{L}_\alpha = \mathbb{1}_{K(\alpha)} \quad \text{as an operator on } \ell^2(P)$$

Words, constructible right ideals, and projections

$\alpha \in \mathcal{W}$; $\dot{\alpha} = e$ — neutral words of even length under concatenation

$K(\alpha) \subset P$ — constructible right ideals under intersection

$\dot{L}_\alpha = \mathbb{1}_{K(\alpha)} \in \mathcal{B}(\ell^2(P))$ — operators in l.r.r. under composition

New idea [X. Li, '12]: only consider isometric representations that respect the $K(\alpha)$.

Definition (X. Li '12)

A representation of P by isometries W_p is *covariant* (Li-covariant) if $e_{K(\alpha)} := \dot{W}_\alpha$ is a s.a. projection that only depends on $K(\alpha)$ for each neutral word α. Automatically, $e_{K(\alpha)} e_{K(\beta)} = e_{K(\alpha) \cap K(\beta)}$.

Definition (X. Li '12)

The *semigroup C*-algebra* $C^*_s(P)$ is the C*-algebra generated by a universal covariant isometric representation $(v_p : p \in P)$.

The C*-algebra $C_s^*(P)$

Since $\dot{L}_\alpha = \mathbb{1}_{K(\alpha)}$, the l.r.r. L_p is Li-covariant, hence $v_p \mapsto L_p$ extends to a homomorphism $\pi_L : C_s^*(P) \to \mathcal{T}_\lambda(P)$.

The restriction of π_L to the diagonal $D_s := C^*(\dot{v}_\alpha : \dot{\alpha} = e)$ sends $\dot{v}_\alpha \mapsto \mathbb{1}_{K(\alpha)}$ and maps D_s onto $D_\lambda = C^*(\mathbb{1}_{K(\alpha)} : \dot{\alpha} = e)$.

However, $D_s \to D_\lambda$ is injective only if the projections $\mathbb{1}_{K(\beta)}$ are linearly independent (X. Li '12).

We say that P *'satisfies independence'* if the projections $\{\mathbb{1}_{K(\beta)} : \dot{\beta} = e\}$ form a linearly independent set.

Equivalently, P satisfies independence if

> *whenever $K(\alpha) = \bigcup_{\beta \in F} K(\beta)$ for some finite set F of neutral words, there is $\beta \in F$ such that $K(\alpha) = K(\beta)$.*

Independence can fail

Example (X. Li '17) Consider the (additive) subsemigroup

$$\Sigma = \mathbb{N}\backslash\{1\} = \{0, 2, 3, \dots\} \ \subset \ \mathbb{Z}.$$

Principal ideals are of the form

$$K(e, n, n, e) = n + \Sigma = \{n, n + 2, n + 3, \dots\}.$$

The constructible (and nonprincipal) ideal

$$K(3, 2, 2, 3) = \Sigma \cap (-3 + 2) + \Sigma = \Sigma \cap (-1) + \Sigma = 2 + \mathbb{N},$$

satisfies

$$2 + \mathbb{N} = (2 + \Sigma) \cup (3 + \Sigma)$$

So independence fails on Σ. By a result of Xin Li,

$$D_s(\Sigma) \text{ and } D_\lambda(\Sigma) \text{ are not canonically isomorphic.}$$

Hence $C_s^*(\Sigma)$ is not a good 'universal model' for $\mathcal{T}_\lambda(\Sigma)$.

A universal Toeplitz C*-algebra

Definition (L-Sehnem '21)

The *universal Toeplitz algebra* $\mathcal{T}_u(P)$ is the universal C*-algebra with generators $\{t_p : p \in P\}$ subject to the relations

(T1) $t_e = 1$;

(T2) $\dot{t}_\alpha = 0$ if $K(\alpha) = \varnothing$ with $\dot{\alpha} = e$;

(T3) $\dot{t}_\alpha = \dot{t}_\beta$ if α and β are neutral words such that $K(\alpha) = K(\beta)$;

(T4) $\prod_{\beta \in F}(\dot{t}_\alpha - \dot{t}_\beta) = 0$ if F is a finite set of neutral words such that $K(\alpha) = \bigcup_{\beta \in F} K(\beta)$ for some neutral word α.

Surprising fact: (T1), (T2), and (T3) force $p \mapsto t_p$ to be an isometric representation of P that satisfies Li-covariance!

In other words, (T1)–(T3) constitute a presentation of $C^*_s(P)$.

(T3), (T4), and independence

Recall the last two relations:

$$\text{(T3)}\ \dot{t}_\alpha = \dot{t}_\beta \ \text{ if } K(\alpha) = K(\beta)$$

$$\text{(T4)}\ \prod_{\beta\in F}(\dot{t}_\alpha - \dot{t}_\beta) = 0 \ \text{ if } K(\alpha) = \bigcup_{\beta\in F} K(\beta)$$

- For all P, (T4) $\implies$ (T3).

Reason: simply take $F = \{\beta\}$.

- If P satisfies independence, (T3) $\implies$ (T4).

Reason: by independence, $K(\alpha) = \bigcup_{\beta\in F} K(\beta)$ can only occur if one of the $K(\beta)$ equals $K(\alpha)$, in which case (T4) holds because the factor $(\dot{t}_\alpha - \dot{t}_\beta)$ vanishes by (T3).

Corollary: If P satisfies independence, then $\mathcal{T}_u(P) = C^*_s(P)$.

A Li-covariant representation that fails (T4)

Let W be the obvious representation of $\Sigma = \mathbb{N}\setminus\{1\} = \{0, 2, 3, \ldots\}$ on $\ell^2(\mathbb{N})$, obtained by taking the l.r.r. of $\mathbb{N}$ and 'throwing away' L_1.

W is Li-covariant and thus gives a representation of $C_s^*(P)$.

Recall the basic failure of independence on Σ:

$K(3,2,2,3) = 2 + \mathbb{N} = (2+\Sigma) \cup (3+\Sigma) = K(e,2,2,e) \cup K(e,3,3,e)$

(T4) would require the product

$$(W_3^* W_2 W_2^* W_3 - W_2 W_2^*)(W_3^* W_2 W_2^* W_3 - W_3 W_3^*)$$

to be zero, but evaluation at $\delta_1 \in \ell^2(\mathbb{N})$ shows it is not 0.

So W fails (T4) and does not give a representation of $\mathcal{T}_u(P)$.

The computation shows directly that $C_s^*(P) \not\cong \mathcal{T}_u(P)$ and also that $D_s(\Sigma) \not\cong D_u(\Sigma)$.

A generalized jointly proper condition

We say that the family $\{W_p : p \in P\}$ is *jointly proper* if

$$\prod_{\alpha \in F} (I - \dot{W}_\alpha) \neq 0$$

for every finite collection $F \subset \mathcal{W}$.

Lemma [L-Sehnem '21]: Suppose $\{W_p : p \in P\}$ satisfies (T1)–(T4). Then

$$t_p \mapsto W_p$$

extends to a C*-algebra homomorphism

$$\mathcal{T}_u(P) \xrightarrow{\pi_W} C^*(W)$$

and $D_u \xrightarrow{\pi_W} D_W$ is an isomorphism iff W is jointly proper.

universal diagonal = reduced diagonal

The left regular representation $(L_p : p \in P)$ is jointly proper, to see this just evaluate projections at δ_e.

Hence
$$D_u \cong D_\lambda.$$

Why is the Lemma about diagonals and not about isomorphic images of $\mathcal{T}_\lambda(P)$?

One reason is the possible failure of amenability (always an issue).

But there is something more remarkable associated to the possible presence of invertible elements $P^* := P \cap P^{-1} \subset P$.

Jointly proper is not enough when $P^* \neq \{e\}$

First recall
Theorem [X. Li '17]: If $P^* = \{e\}$, then a given representation of $\mathcal{T}_\lambda(P)$ is faithful iff it is faithful on D_λ.

Corollary: If $P^* = \{e\}$, then a representation of $\mathcal{T}_\lambda(P)$ is faithful iff it is jointly proper.

What if the group $P^* = P \cap P^{-1}$ of units in P is nontrivial?

Take $P = P^* = G$. Then there are no proper constructible right ideals, so every representation is jointly proper (vacuously).

We still have $D_u = D_\lambda = \mathbb{C}I$, but e.g. if $G \neq \{e\}$ is abelian, any character gives a proper quotient of $\mathcal{T}_\lambda(P) = C^*(G)$.

Hence we need something extra when $P^* \neq \{e\}$.

The action of P^* on D_λ

For each $u \in P^*$, L_u is unitary and there is an action γ of P^* on $D_\lambda = C^*(\dot{L}_\alpha : \alpha \in \mathcal{W})$ given by

$$\gamma_u(\dot{L}_\alpha) = \dot{L}_{(e,u,\alpha,u,e)} = L_u \dot{L}_\alpha L_u^*.$$

In fact, γ is the restriction of a partial action of G on D_λ and

$$D_\lambda \rtimes_{\gamma,r} P^* \quad \subset \quad D_\lambda \rtimes_{\gamma,r} G \;\cong\; \mathcal{T}_\lambda(P)$$

i.e. $D_\lambda \rtimes_{\gamma,r} P^*$ embeds canonically in $\mathcal{T}_\lambda(P)$.

Theorem [L-Sehnem '21]: The subalgebra $D_\lambda \rtimes_{\gamma,r} P^*$ has nontrivial intersection with every nontrivial ideal of $\mathcal{T}_\lambda(P)$.

Equivalently:
a representation of $\mathcal{T}_\lambda(P)$ is faithful iff it is faithful on $D_\lambda \rtimes_{\gamma,r} P^*$.

Idea of the proof

Key step: if ρ is a representation of $\mathcal{T}_\lambda(P)$ that is faithful on $D_\lambda \rtimes_{\gamma,r} P^*$, then there is a conditional expectation Φ_ρ that completes the commutative diagram.

$$\begin{tikzcd}
\mathcal{T}_\lambda(P) \arrow[r, "\rho"] \arrow[d, "\Phi_\lambda"] & C^*(\rho) \arrow[d, "\Phi_\rho"] \\
D_\lambda \arrow[r, "\rho|_{D_\lambda}"] & D_\rho
\end{tikzcd}$$

and the bottom horizontal arrow is an isomorphism.

The construction of Φ_ρ is an adaptation for general submonoids of groups of the direct argument given for QLO groups from [L-Raeburn '96], which, borrows itself heavily from Cuntz's proof of uniqueness of $\mathcal{O}_n$.

A general uniqueness theorem

If we assume weak containment, the faithfulness result can be reformulated as a universality + uniqueness theorem for $\mathcal{T}_\lambda(P)$.

Theorem [L-Sehnem]: Suppose the conditional expectation $\Phi_u : \mathcal{T}_u(P) \to D_u$ is faithful, and let $(W_p : p \in P)$ be a collection of elements satisfying (T1)–(T4).

Then the map $L_p \mapsto W_p$ extends to a C*-algebra homomorphism

$$\mathcal{T}_u(P) \xrightarrow{\pi_W} C^*(W)$$

and π_W is an isomorphism iff its restriction to $D_\lambda \rtimes_{\gamma,r} P^*$ is an isomorphism.

For a class of monoids P it is possible to decide uniqueness based solely on joint properness of W.

Universality and uniqueness for (TF) monoids

We will say that the monoid $P \subset G$ satisfies condition (TF) if for every $u \in P^* \backslash \{e\}$ and every finite collection $\mathcal{C}$ of proper constructible right ideals, there exists $t \in P \backslash \bigcup_{R \in \mathcal{C}} R$ such that $ut \notin tP^*$. (equiv. $utP \neq tP$)

Theorem (L-Sehnem '21)

Suppose $P \subset G$ satisfies (TF) and $\Phi_u : \mathcal{T}_u(P) \to D_u$ is faithful. Let $(W_p : p \in P)$ be a collection of elements satisfying (T1)–(T4).

Then the map $L_p \mapsto W_p$ extends to homomorphism

$$\mathcal{T}_\lambda(P) \xrightarrow{\pi_W} C^*(W)$$

and π_W is an isomorphism if and only if W is jointly proper.

Sketch of proof

Since Φ_u is faithful,

$$\pi_L : \mathcal{T}_u(P) \xrightarrow{\cong} \mathcal{T}_\lambda(P)$$

so $\mathcal{T}_\lambda(P)$ is universal and π_W exists.

It suffices to show that when P is (TF) and W is jointly proper, then π_W is faithful on $D_\lambda \rtimes_{\gamma,r} P^*$.

By Archbold–Spielberg, building upon work of Tomiyama:

> *representations of $D_\lambda \rtimes_{\gamma,r} P^*$ are faithful whenever they are faithful on $D_r \iff P^* \circlearrowright D_r$ is topologically free*

Condition (TF) precisely characterizes semigroups P for which $P^* \circlearrowright D_r$ is topologically free.

And from before, π_W is faithful on D_r iff W is jointly proper.

$\square$

Relative orthogonal complements

Emerging Moral:

> *The relevant information about families isometries modeling the l.r.r. seems to reside in the interrelations among the* relative orthogonal complements *of their range projections and of the projections associated to constructible right ideals.*

We have seen two such instances: (T4) and the joint properness condition.

Next we'll briefly sketch another construction that relies on these complements to give a presentation of Sehnem's covariance algebra and show that it constitutes an appropriate universal version of the boundary quotient for $\mathcal{T}_u(P)$.

The (reduced) boundary quotient of $\mathcal{T}_\lambda(P)$

Recall first Xin Li's definition of the boundary quotient of $\mathcal{T}_\lambda(P)$, generalizing that for QLO from [Crisp-L '07].

For $P \subset G$, Xin Li realized $\mathcal{T}_\lambda(P)$ as the reduced crossed product

$$\mathcal{T}_\lambda(P) = C(\Omega) \rtimes_r G$$

of a partial action of G on $\Omega = \sigma(D_\lambda)$.

As it turns out, Ω always has a smallest closed invariant subset $\partial\Omega$ and the (reduced) boundary quotient is

$$\partial\mathcal{T}_\lambda(P) := C(\partial\Omega) \rtimes_r G.$$

Strong covariance and universal boundaries

For a boundary quotient (defined in terms of a universal property) we may take

1. The full partial crossed product $C(\partial\Omega) \rtimes G$.

This is a special case of a more general construction

2. Sehnem's strong covariance algebra for the canonical product system with one dimensional fibers over $P \subset G$,

These ought to be quotients of $\mathcal{T}_u(P)$.

Indeed, it is possible to write extra relations which when added to (T1)–(T4) give Sehnem's *strong covariance ideal* and providing a simplified presentation of the strong covariance algebra in terms of generators and relations.

Foundation sets and the covariance algebra

Definition: Let S be a constructible right ideal of P. We say that a finite collection $\mathcal{C}$ of constructible ideals is a *foundation set* for S, if $R \subset S$ for all $R \in \mathcal{C}$ and for all $p \in S$, one has

$$pP \cap \Big(\bigcup_{R \in \mathcal{C}} R \Big) \neq \varnothing.$$

$\mathcal{C}$ is *proper* if $S \backslash \big(\bigcup_{R \in \mathcal{C}} R \big) \neq \varnothing$. So that (T4) does not apply at S!

Boundary relations:

$$\prod_{\beta \in A} (\dot{w}_\alpha - \dot{w}_\beta) = 0$$

for neutral α and proper foundation set $\{K(\beta) \mid \beta \in A\}$ for $K(\alpha)$. Note: The relations corresponding to foundation sets that are not proper already fall within the original (T1)–(T4).

Theorem: The covariance algebra is the quotient of $\mathcal{T}_u(P)$ by the boundary relations. (several other characterizations are also given)

Purely infinite simple boundary quotients

We say that $P \subset G$ satisfies condition (PI) provided that

for all $p, t \in P$ with $p \neq t$ there is $s \in P$ such that $psP \cap tsP = \emptyset$.

Theorem (L-Sehnem '21)

Let $\{e\} \neq P \subset G$. If P satisfies (PI) then $\partial\mathcal{T}_\lambda(P)$ is purely infinite simple. The converse implication also holds if the boundary action is amenable, in the sense that $\mathbb{C} \rtimes_{\mathbb{C}^P} P \cong \partial\mathcal{T}_\lambda(P)$ via the canonical map Λ_∂.

Thanks once again!