Δυναμικά συστήματα, ημι-σταυρωτά γινόμενα και το ριζικό τους

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The Covariance Algebra of a Dynamical System (X, ϕ)

X: locally compact Hausdorff [metrisable]
$\phi : X \to X$ continuous, onto,
proper (i.e. K cpct $\Rightarrow \phi^{-1}(K)$ cpct).
$\alpha : C_0(X) \to C_0(X) : f \to f \circ \phi$ isometric *-endomorphism.
The linear space

$$c_{00}(\mathbb{Z}_+, C_0(X)) = c_{00}(\mathbb{Z}_+) \otimes C_0(X)$$

consists of formal 'polynomials'

$$a = (f_n) = \sum_{n=0}^{N} e_n \otimes f_n = \sum_{n \geq 0} u^n f_n \,, \qquad f_n \in C_0(X) \,.$$

A multiplication on $c_{00}(\mathbb{Z}_+, C_0(X))$ is defined by setting

$$u^n f \, u^m g = u^{m+n} (\alpha^m(f) g)$$

and extending by linearity:

$$(\sum_n u^n f_n) * (\sum_m u^m g_m) = \sum_k u^k (\sum_m \alpha^m(f_{k-m}) g_m) \,.$$

Covariant representation

May assume $C_0(X) \simeq \mathcal{C} \subseteq \mathcal{B}(H_0)$.
So each $f \in C_0(X)$ is identified with an operator on H_0.

Idea: 'Enlarge' the space (if necessary) to accomodate U and $\pi(f)$ on H so that $\pi(\alpha(f)) = U^*\pi(f)U \;\; \forall f \in \mathcal{C}$ holds:

Consider

$$H = \ell^2(\mathbb{Z}_+) \otimes H_0 := \{(\xi(n))_{n\in\mathbb{Z}_+} : \xi(n) \in H_0 \; \forall n, \; \sum_n \|\xi(n)\|^2_{H_0} < \infty\}$$

$$\langle(\xi(n)), (\eta(n))\rangle := \sum_n \langle \xi(n), \eta(n)\rangle_{H_0}$$

and $U_0 : H \to H$:

$$U_0 : (\xi(0), \xi(1), \xi(2), \dots) \to (0, \xi(0), \xi(1), \dots)$$

Covariant representation

Notation: for $n \in \mathbb{Z}_+$ and $\xi \in H_0$ denote by $e_n \otimes \xi \in H$ the function

$$\mathbb{Z}_+ \to H_0 : m \to (e_n \otimes \xi)(m) = \begin{cases} \xi, & m = n \\ 0, & m \neq n \end{cases}$$

(note $H = \overline{\text{span}}\{e_n \otimes \xi : n \in \mathbb{Z}_+, \xi \in H_0\}$).
The map U_0 is given by

$$U_0(e_n \otimes \xi) = e_{n+1} \otimes \xi.$$

Also define the representation $\pi_0 : \mathcal{C} \to \mathcal{B}(H)$ by

$$\pi_0(f)(e_n \otimes \xi) = e_n \otimes \alpha^n(f)\xi$$

where $f \in \mathcal{C}, \xi \in H_0, n \in \mathbb{Z}_+$.

Covariant representation

Representing these as matrices with entries in $\mathcal{B}(H_0)$,

$$\pi_0(f) = \operatorname{diag}(\alpha^n(f)) = \begin{bmatrix} f & & \\ & \alpha(f) & \\ & & \ddots \end{bmatrix},$$

$$U_0 = \begin{bmatrix} 0 & & \\ \mathbf{1}_{H_0} & 0 & \\ & \mathbf{1}_{H_0} & \ddots \\ & & \ddots \end{bmatrix}.$$

$\|\pi_0(f)\| = \|f\|_{C_0(X)}$
U_0: (proper) isometry.

Covariant representation

We have

$$\pi_0(f)U_0 : e_n \otimes \xi \xrightarrow{U_0} e_{n+1} \otimes \xi \xrightarrow{\pi_0(f)} e_{n+1} \otimes \alpha^{n+1}(f)\xi$$

$$U_0\pi_0(\alpha(f)) : e_n \otimes \xi \xrightarrow{\pi_0(\alpha(f))} e_n \otimes \alpha^n(\alpha(f))\xi \xrightarrow{U_0} e_{n+1} \otimes \alpha^n(\alpha(f))\xi$$

hence

$$\pi_0(f)U_0 = U_0\pi_0(\alpha(f)), \quad \textbf{equivalently} \qquad \pi_0(\alpha(f)) = U_0^*\pi_0(f)U_0.$$

Now define

$$\pi := U_0 \times \pi_0 : c_{00}(\mathbb{Z}_+) \otimes \mathcal{C} \to \mathcal{B}(H)$$
$$\pi\left(\sum_k u^k f_k\right) = \sum_k U_0^k \pi_0(f_k)\,.$$

Covariant representation

Πρόταση

The representation $U_0 \times \pi_0$ just constructed is injective on the covariance algebra $c_{00}(\mathbb{Z}_+) \otimes \mathcal{C}$.

Indeed, suppose $(U_0 \times \pi_0)\left(\sum_k u^k f_k\right) = 0$, i.e. $\sum_k U_0^k \pi_0(f_k) = 0$. Then for all $\xi, \eta \in H_0$ and all $m \in \mathbb{Z}_+$ we have

$$0 = \sum_{k=0}^{\infty} U_0^k \pi_0(f_k)(e_0 \otimes \xi) = \sum_k U_0^k(e_0 \otimes \alpha^0(f_k)\xi) = \sum_k e_k \otimes \alpha^0(f_k)\xi$$

$$\text{and so} \quad 0 = \left\langle \sum_k e_k \otimes f_k \xi, e_m \otimes \eta \right\rangle = \langle f_m \xi, \eta \rangle_{H_0}$$

which shows that $f_m = 0$ and so, since m is arbitrary, that $\sum_k u^k f_k = 0$ in $c_{00}(\mathbb{Z}_+) \otimes \mathcal{C}$. □

The Semicrossed product $\mathcal{C} \rtimes_\alpha \mathbb{Z}_+$

This is defined to be the closure of the covariance algebra $\mathcal{A}_0 := c_{00}(\mathbb{Z}_+) \otimes \mathcal{C}$ in the norm induced by the (injective) representation π.
The norm on $\mathcal{A}_0$ is given by

$$\left\| \sum_k u^k f_k \right\| := \left\| (U_0 \times \pi_0) \left(\sum_k u^k f_k \right) \right\|_{\mathcal{B}(H)} = \left\| \sum_k U_0^k \pi_0(f_k) \right\|_{\mathcal{B}(H)}$$

So, $\mathcal{A} := \mathcal{C} \rtimes_\alpha \mathbb{Z}_+$ is a (norm-closed) operator algebra which is non-selfadjoint.
It is concretely represented as block-lower triangular operators on $H \simeq \bigoplus_{n=0}^{\infty} H_0$.
In fact, its 'diagonal' $\mathcal{A} \cap \mathcal{A}^*$ is just $\pi(\mathcal{C})$.

Fourier coefficients

For $k \in \mathbb{Z}_+$, define

$$E_k : \mathcal{A}_0 \to \mathcal{C} : a = \sum_n u^n f_n \to f_k.$$

Clearly linear. Also $\|\cdot\|$-contractive: for $\xi, \eta \in H_0$ of norm one,

$$\langle f_m \xi, \eta \rangle_{H_0} = \left\langle \sum_k U_0^k \pi_0(f_k)(e_0 \otimes \xi), (e_m \otimes \eta) \right\rangle_H$$

$$\Rightarrow \left| \langle f_m \xi, \eta \rangle_{H_0} \right| \leq \left\| \sum_k U_0^k \pi_0(f_k) \right\|_{\mathcal{B}(H)} = \left\| \sum_n u^n f_n \right\|$$

$$\Rightarrow \|E_m(a)\|_{\mathcal{C}} = \|f_m\|_{\mathcal{C}} \leq \|a\| \quad \forall a \in \mathcal{A}_0$$

Fourier coefficients

Thus, for all $k \in \mathbb{Z}_+$, the map E_k extends by continuity to a linear contraction on the $\|\cdot\|$-completion:

$$E_k : \mathcal{C} \rtimes_\alpha \mathbb{Z}_+ \to \mathcal{C}$$

How to isolate the k-th Fourier coefficient of an arbitrary $a \in \mathcal{A}$?

Locating $E_k(a)$

Define *the gauge action* (of $\mathbb{T}$) first on $\mathcal{A}_0$: for $e^{it} \in \mathbb{T}$, let

$$\theta_t\left(\sum_n u^n f_n\right) = \sum_n (e^{it}u)^n f_n$$

Claim. Each θ_t extends to an isometric automorphism of $\mathcal{C} \rtimes_\alpha \mathbb{Z}_+$.

Proof For each $e^{it} \in \mathbb{T}$ let $V_t(e_m \otimes \xi) \mapsto e^{itm}(e_m \otimes \xi)$: this extends to an isometry $V_t : H \to H$, clearly onto. Can verify that $\theta_t(a) = V_t a V_t^*$. Hence $\|\theta_t(a)\| \leq \|a\|$ when $a \in \mathcal{A}_0$. Etc. □

Thus θ defines an action of the group $\mathbb{T}$ on $\mathcal{C} \rtimes_\alpha \mathbb{Z}_+$.

Now we calculate, first when $a \in \mathcal{A}_0$ and then for general $a \in \mathcal{A}$, (since $t \mapsto \theta_t(a)$ is continuous, so $\mathcal{A}_0$-valued integral exists)

$$\frac{1}{2\pi}\int_0^{2\pi} \theta_t(a)e^{-imt}dt = u^m E_m(a).$$

Finally...

Πρόταση

Each $a \in \mathcal{C} \rtimes_\alpha \mathbb{Z}_+$ belongs to the $\|\cdot\|$-closed linear span of

$$\{u^k E_k(a) : k \in \mathbb{Z}_+\}.$$

Write $a \sim \sum_n u^n E_n(a)$. In fact, there is a Fejér expansion:

$$\lim_N \left\| a - \sum_{n=0}^{N} \left(1 - \frac{n}{N+1}\right) u^n E_n(a) \right\| = 0.$$

So, if all $E_k(a)$ vanish, then a must vanish. Moreover,

Πρόταση

If $J \subseteq \mathcal{A}$ is a closed ideal, invariant under the gauge automorphisms, then $a \sim \sum_n u^n E_n(a) \in \mathcal{A}$ belongs to J iff each monomial $u^n E_n(a)$ belongs to J.

The Radical

Let $\mathcal{A}$ be an algebra. The Jacobson Radical of $\mathcal{A}$ can be defined by

$$\text{Rad}\mathcal{A} = \{q \in \mathcal{A} : \sigma_{\mathcal{A}}((\lambda + a)q) = 0 \text{ for all } a \in \mathcal{A} \text{ and } \lambda \in \mathbb{C}\}.$$

For a Banach algebra,

$$\text{Rad}\mathcal{A} = \{q \in \mathcal{A} : (\lambda + a)q \text{ is quasinilpotent for all } a \in \mathcal{A} \text{ and } \lambda \in \mathbb{C}\}$$

where $x \in \mathcal{A}$ is called quasinilpotent if $\lim \|x^n\|^{1/n} = 0$.

Πρόταση

Let $\mathcal{A} = \mathcal{C} \times_\alpha \mathbb{Z}^+$. An element $a \in \mathcal{A}$ is in the radical of $\mathcal{A}$ iff $u^n E_n(a) \in \text{Rad}\mathcal{A}$ for all $n \geq 0$. In particular all elements of the radical satisfy $E_0(a) = 0$. (They are strictly lower-triangular).

Wandering points give elements in the radical

Recall the dynamical system (X, ϕ).

Say that a set $A \subseteq X$ is wandering if the sets $\phi^{-n}(A)$ ($n \geq 0$) are pairwise disjoint. Say that a point $x \in X$ is wandering if it has a wandering neighbourhood.

Θεώρημα (Muhly)

If (X, ϕ) *has a wandering point then* $\mathrm{Rad}(\mathcal{C} \times_\alpha \mathbb{Z}^+) \neq \{0\}$.

Indeed (M. Anoussis) if $f \in \mathcal{C}$ is compactly supported in a wandering open set then $uf \in \mathcal{A}$ generates a nonzero nilpotent ideal, hence is in $\mathrm{Rad}(\mathcal{A})$.

Recurrent points give elements outside the radical

Ένα σημείο $x \in X$ καλείται επανερχόμενο (recurrent) για το δυναμικό σύστημα (X, ϕ) αν για κάθε περιοχή U του x υπάρχει $n \geq 1$ ώστε $\phi^n(x) \in U$.

Πρόταση (Κρίσιμο Λήμμα)

Έστω $f \in C_0(X)$ και x επανερχόμενο σημείο του (X, ϕ). Αν $f(x) \neq 0$, τότε $u^l f \notin \mathrm{Rad}(\mathcal{A})$, για κάθε $l \in \mathbb{Z}^+$.

Θεώρημα

Αν $X_r \subseteq X$ είναι το σύνολο των επανερχομένων σημείων του (X, ϕ),

$$\mathrm{Rad}(\mathcal{A}) = \left\{a \sim \sum_{n\geq 1} u^n f_n : f_n|_{X_r} = 0 \ \forall n\right\}.$$