# Representation rigidity of subgroups and ideal structure of C*-algebras of quasi-regular representations

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Throughout the talk $G$ is a countable discrete group

We consider unitary representations of $G$, that is, group homomorphisms $\pi : G \to \mathcal{U}(H_\pi)$, where $H_\pi$ is a separable Hilbert space, and $\mathcal{U}(H)$ is the set of all unitary operators on $H$.
We denote by $\mathrm{Rep}(G)$ the collection of all unitary reps of $G$.

$K \leq H_\pi$ is ($G$-)invariant if $\pi(g)K \subset K$ for all $g \in G$
$\rightsquigarrow \sigma(\cdot) = p_K \pi(\cdot) p_K$ is a unitary rep on $K$
In this case we say $\pi$ contains $\sigma$ and write $\sigma \subset \pi$

$\pi$ is irreducible if $\nexists$ inv subspace except 0 and $H_\pi$.

Two reps $\pi$ and $\sigma$ are *unitary equivalent*, denoted $\pi \sim_u \sigma$ if $\exists$ unitary $u : H_\pi \to H_\sigma$ such that $u\pi(g)u^* = \sigma(g)$ $\forall g \in G$.

## Examples

- The trivial rep $1_G$; $1_G(g) = I_H$ for all $g \in G$
  obviously irreducible

- The regular rep $\lambda_G : G \to \mathcal{U}(\ell^2(G))$

  $$[\lambda_G(g)\xi](k) = \xi(g^{-1}k) \qquad (g, k \in G,\ \xi \in \ell^2(G))$$

  never irreducible (unless trivial)

## Examples

More generally:
The quasi-regular rep for $H \leq G$

$$\lambda_{G/H} : G \to \mathcal{U}(\ell^2(G/H))$$

$$[\lambda_{G/H}(g)\,\xi]\,(kH) = \xi(g^{-1}kH) \qquad (g, k \in G,\ \xi \in \ell^2(G/H))$$

General Goal: study the map

$$\mathrm{Sub}(G) \ni H \ \longmapsto\ \lambda_{G/H} \in \mathrm{Rep}(G)$$

and understand the ideal structure of the C*-algebra
$C^*_{\lambda_{G/H}}(G) = \overline{\mathrm{span}}\{\lambda_{G/H}(g) : g \in G\} \subset B(\ell^2(G/H))$

Question (Rigidity): To what extent $\lambda_{G/H}$ determines $H$?

We like to restrict to irreducible representations. We have the following characterization due to Mackey:

$\lambda_{G/H}$ is irreducible iff $H$ is *self-commensurating* (s.c.) in $G$, that is $\{g \in G : [H : H \cap gHg^{-1}] < \infty \text{ and } [gHg^{-1} : H \cap gHg^{-1}] < \infty\} = H$

Theorem (Mackey 51):
Let $H, L \in Sub(G)$ be s.c. Then

$$\lambda_{G/H} \sim_u \lambda_{G/L} \iff H \sim_{\mathrm{conj}} L$$

But, unitary equivalence is too restrictive in general for rep of non-compact groups, and specially in the case of discrete groups

e.g. the classification of $\widehat{G} = \mathrm{Irr(G)} / \sim_u$ is hopeless, unless $G$ is virtually abelian (Glimm and Thoma)

The more appropriate notion of equivalence (and inclusion) for reps of discrete groups is that of *weak equivalence*

Prim($G$): the set of irreducible unitary representations of $G$ up to weak equivalence is a more accessible dual space of $G$ (e.g. always a standard Borel space)

## Weak containment

For $\pi, \sigma \in \mathrm{Rep}(G)$ we say $\sigma$ is *weakly contained* in $\pi$, denoted $\sigma \prec \pi$ iff

$$\Big\|\sum_j c_j \sigma(g_j)\Big\|_{B(H_\sigma)} \;\leq\; \Big\|\sum_j c_j \pi(g_j)\Big\|_{B(H_\pi)}$$

for any finite sequence $c_1, c_2, \ldots \in \mathbb{C}$; equivalently, the map $\pi(g) \mapsto \sigma(g)$ extends to a C*-homomorphism $C^*_\pi(G) \to C^*_\sigma(G)$

Thus, weak containment structure of $\pi$ is equivalent to ideal structure of its C*-algebra $C^*_\pi(G)$

We say $\pi$ and $\sigma$ are weakly equivalent, denoted $\sigma \sim_w \pi$, iff $\sigma \prec \pi$ and $\pi \prec \sigma$

Example:

$1_G \subset \lambda_G$ iff $G$ is finite, and $1_G \prec \lambda_G$ iff $G$ is amenable

Recall: $G$ is amenable if there is a non-zero positive linear functional $M : \ell^\infty(G) \to \mathbb{C}$ that is $G$-invariant: $M(\varphi_g) = M(\varphi)$ for all $\varphi \in \ell^\infty(G)$ and $g \in G$, where $\varphi_g(h) = \varphi(g^{-1}h)$

In the case of quasi-regular reps:

$\lambda_{G/H} \subset \lambda_G$ iff $H$ is finite, and $\lambda_{G/H} \prec \lambda_G$ iff $H$ is amenable

Mackey's rigidity result fails for weak equivalence:
Let $G = \mathbb{F}_2 =< a, b >$ be the free group on two generators. Then $H =< a >$ and $L =< b >$ are both s.c., and $\lambda_{G/H} \sim_w \lambda_{G/L}$ but $H \not\sim_{\mathrm{conj}} L$

But, there are rigid examples: $G = H * H$ with $H$ non-amenable (e.g. the free group $\mathbb{F}_2$). Then $H$ is s.c. in $G$ and for any s.c. $L \leq G$, we have $\lambda_{G/H} \sim_w \lambda_{G/L}$ iff $H \sim_{\mathrm{conj}} L$

Theorem:

Let $H$ be a subgroup of $G$ with the *spectral gap property*. Then $H$ is s.c. and for any s.c. $L \leq G$, we have $\lambda_{G/H} \sim_w \lambda_{G/L}$ iff $H \sim_{\mathrm{conj}} L$

Definition:

$H \leq G$ has the *spectral gap property* (s.g.) if there is no $H$-invariant mean on $\ell^\infty(G/H \setminus \{H\})$.

## Examples of s.g. subgroups

Theorem:
Let $H \leq G$ be non-amenable such that $H \cap gHg^{-1}$ is amenable for every $g \in G - H$. Then $H \in \mathrm{Sub}_{\mathrm{sg}}(G)$.

Example:
$SL_2(\mathbb{Z}) \leq SL_3(\mathbb{Z})$

Example:
Let $G = H * L$ (non-trivial free product).
If $H$ is non-amenable then $H$ has s.g.
In particular, $\lambda_{G/H} \sim_w \lambda_{G/L}$ iff both $H$ and $L$ are amenable. (cf. $H = L = \mathbb{Z}$).

## Examples of s.g. subgroups

Theorem:
If $H \in \mathrm{Sub}(G)$ is "strongly s.c." with property (T), then $H \in \mathrm{Sub}_{\mathrm{sg}}(G)$.

Example:
$SL_n(\mathbb{Z}) \leq SL_{n+1}(\mathbb{Z})$, $n > 2$.

## On ideal structure of $C^*_{\lambda_{G/H}}(G)$

Theorem:

Let $H \in \mathrm{Sub}_{\mathrm{sg}}(G)$. Then $C^*_{\lambda_{G/H}}(G)$ has a smallest non-zero ideal $I_{\min}$ (i.e. contained in every ideal of $C^*_{\lambda_{G/H}}(G)$).

## On ideal structure of $C^*_{\lambda_{G/H}}(G)$

Back to the map

$$\mathrm{Sub}(G) \ni H \longmapsto \lambda_{G/H} \in \mathrm{Rep}(G)$$

## On ideal structure of $C^*_{\lambda_{G/H}}(G)$

Back to the map

$$\frac{\mathrm{Sub}(G)}{\sim_{\mathrm{conj}}} \ni H \longmapsto \lambda_{G/H} \in \frac{\mathrm{Rep}(G)}{\sim_w}$$

$\mathrm{Sub}(G) \subset 2^G \rightsquigarrow$ a compact space
$G \curvearrowright \mathrm{Sub}(G)$: $g \cdot H = gHg^{-1}$

$\mathrm{Rep}(G)$ carries the Fell topology: $\pi_n \to \pi$ iff $\pi \prec \oplus_k \pi_{n_k}$ for every subsequence $(\pi_{n_k})_k$ of $(\pi_n)_n$

Theorem (Fell):
The above map is continuous.

In particular, if $g_n H g_n^{-1} \to \{e\}$, then $\lambda_G \prec \lambda_{G/H}$

# On ideal structure of $C^*_{\lambda_{G/H}}(G)$

Definition (Furstenberg):
$G \curvearrowright X$ is a *boundary action* if $\forall x \in X$ and $\forall \nu \in \mathrm{Prob}(X)$ there is a net $(g_i) \in G$ such that $g_i \nu \xrightarrow{\text{weak*}} \delta_x$.

Example:
$\mathbb{F}_2 \curvearrowright \partial \mathbb{F}_2$

Example:
$SL_n(\mathbb{Z}) \curvearrowright \mathbb{P}^{n-1}(\mathbb{R})$

Example:
$G$ is amenable iff $G$ has no non-trivial boundary action

## On ideal structure of $C^*_{\lambda_{G/H}}(G)$

Definition:

We say $H \leq G$ is *weakly parabolic* if $\exists$ top. free boundary action $G \curvearrowright X$ such that $\mathrm{Prob}_H(X) \neq \emptyset$

Theorem:

Let $H \leq G$ be weakly parabolic. Then $C^*_{\lambda_{G/H}}(G)$ contains a largest proper ideal $I_{\max}$ (i.e. contains every ideal of $C^*_\pi(G)$) Moreover,

$$\frac{C^*_{\lambda_{G/H}}(G)}{I_{\max}} \cong C^*_{\lambda_G}(G)$$

## On ideal structure of $C^*_{\lambda_{G/H}}(G)$

Example:

Let $G$ be a group which admits a top. free boundary action. Then every amenable $H \leq G$ is weakly parabolic.

Example:

$G = PGL_n(\mathbb{Q})$ for $n \geq 2$, $H = PO_n(\mathbb{Q})$

Example:

$G = PSL_n(\mathbb{Q})$ for $n \geq 2$, $H = PSL_n(\mathbb{Z})$

## Weakly parabolic subgroups with spectral gap property

Corollary:
Let $H \leq G$ be a weakly parabolic subgroup with the spectral gap property. Then there are ideals $I_{\min}, I_{\max} \leq C^*_{\lambda_{G/H}}(G)$ such that

$$0 \neq I_{\min} \leq I \leq I_{\max} \lneq C^*_{\lambda_{G/H}}(G)$$

for every non-zero proper ideal $I$.

Example:
$G = PSL_n(\mathbb{Z})$, $H = PSL_{n-1}(\mathbb{Z})$ for $n \geq 3$

Example:
$G = PGL_n(\mathbb{Q})$, $H = PGL_{n-1}(\mathbb{Q})$, for $n \geq 3$

Thank You!