# Beurling-Fourier Algebras and Complexification

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# Complexification of a Group: Abelian case

Let $G$ be an Abelian locally compact group. The dual group $\hat{G}$ is the set of continuous homomorphisms:

$$\chi : G \mapsto \mathbb{T}.$$

The Pontryagin duality: $\hat{\hat{G}} = G$.

**Definition**

We define the Abelian complexification $G^a_{\mathbb{C}}$ as the set of continuous homomorphisms

$$\varphi : \hat{G} \to \mathbb{C} \setminus \{0\}.$$

We have $\mathbb{R}^a_{\mathbb{C}} \simeq \mathbb{C}$, $\mathbb{T}^a_{\mathbb{C}} \simeq \mathbb{C} \setminus \{0\}$, $\mathbb{Z}^a_{\mathbb{C}} \simeq \mathbb{Z}$.

# Complexification of a compact group

Let $G$ be a *compact group* and $x \mapsto \lambda(x)$ the left regular representation of $G$, $\lambda(x)\xi(y) = \xi(x^{-1}y)$, $\xi \in L^2(G)$. Consider $VN(G) = \{\lambda(x) : x \in G\}''$.
If $G$ is compact, and $\hat{G}$ is the unitary dual of $G$, treated as a set of unitary representations $\pi : G \to \mathcal{U}(H_\pi)$, $d_\pi = \dim(H_\pi)$, we have that

$$\lambda \simeq \oplus_{\pi\in\hat{G}}\pi \text{ and } VN(G) \simeq \oplus^{\ell^\infty}_{\pi\in\hat{G}}\mathcal{L}(H_\pi).$$

Let $\mathrm{Trig}(G)$ be the span of matrix coefficients of elements in $\hat{G}$:

$$\mathrm{Trig}(G) = \oplus_{\pi\in\hat{G}}\mathrm{Trig}_\pi(G).$$

The Fourier transform: $u \in \mathrm{Trig}(G) \mapsto (\hat{u}(\pi))_{\pi\in\hat{G}}$ where

$$\hat{u}(\pi) = \int_G u(s)\pi(s^{-1})ds \in \mathcal{L}(H_\pi).$$

Hence the linear dual space $\mathrm{Trig}(G)^\dagger$ can be identified with $\prod_{\pi\in\hat{G}}\mathcal{L}(H_\pi)$ via

$$\langle u, (T_\pi)\rangle = \sum_{\pi\in\hat{G}} d_\pi \mathrm{Tr}(\hat{u}(\pi)T_\pi).$$

*Note that $\prod_{\pi\in\hat{G}}\mathcal{L}(H_\pi)$ is the set of affiliated elements with $VN(G)$.*

## Compact groups complexification, cont.

**Definition [McKennon '79, Cartwright-McMullen, '81]**

We define a complexification of $G_{\mathbb{C}}$ of a compact group $G$ as the set of characters of $\mathrm{Trig}(G)$, i.e.

$$\mathbb{G}_{\mathbb{C}} = \{T \in \prod_{\pi \in \hat{G}} \mathcal{L}(H_\pi) \setminus \{0\} : \langle T, uu' \rangle = \langle T, u \rangle \langle T, u' \rangle, u, u' \in Trig(G)\}$$

$T \in G_{\mathbb{C}}$ iff $m^\dagger(T) = T \otimes T$, where $m$ is the multiplication on $\mathrm{Trig}(G)$.
We identify $G \simeq \{(\pi(s))_{\pi \in \hat{G}} : s \in G\} = \lambda(G)$.

**Theorem (Krein, Tanaka, McKennon, Cartwright, McMullen)**

- *$G_{\mathbb{C}}$ is a group*
- $G_{\mathbb{C}} \cap VN(G) = \lambda(G) \simeq G$
- $T \in G_{\mathbb{C}} \Rightarrow |T| = (|T_\pi|)_{\pi \in \hat{G}} \in G_{\mathbb{C}}$ *and* $|T|^{-1} T \in G$.

Cartan decomposition: $G_{\mathbb{C}} = G \cdot G_{\mathbb{C}}^{+}$.

# WHY COMPLEXIFICATION?

Consider

$$\mathfrak{g}_{\mathbb{C}} = \{X \in \prod_{\pi \in \hat{G}} \mathcal{L}(H_\pi) : \langle X, uu' \rangle = \langle X, u \rangle u'(e) + u(e) \langle X, u' \rangle, u, u' \in \mathrm{Trig}(G)\},$$

i.e. $m^\dagger(X) = X \otimes I + I \otimes X$,

$$\mathfrak{g} = \{X \in \mathfrak{g}_{\mathbb{C}} : X = -X^*\} \; \mathfrak{g}_{\mathbb{C}} = \mathfrak{g} + i\mathfrak{g}$$

$\mathfrak{g}_{\mathbb{C}}$ is a Lie algebra, $\mathfrak{g}$ is a real Lie subalgebra.

## Theorem (Cartwright, McMullen)

- $\exp(tX) \in G_{\mathbb{C}} \Leftrightarrow X \in \mathfrak{g}_{\mathbb{C}}$; $\exp(tX) \in G \Leftrightarrow X \in \mathfrak{g}$;
- $\exp(i\mathfrak{g}) = G_{\mathbb{C}}^{+}$, $\exp(\mathfrak{g}) \subset G$.

If $G$ is a connected Lie group with the Lie algebra $\mathfrak{g}$, then $G_{\mathbb{C}}$ is a Lie group with Lie algebra $\mathfrak{g}_{\mathbb{C}}$.

## Example

$SU(n)_{\mathbb{C}} = SL(n, \mathbb{C})$, $\mathbb{T}_{\mathbb{C}} = \mathbb{C} \setminus \{0\}$. If $G$ is abelian and compact then $G_{\mathbb{C}}^{a} \simeq G_{\mathbb{C}}$.

# Complexification: general case

Let $G$ be a locally compact group $G$, $\lambda$ the left regular representation, $VN(G)$ the group von Neumann algebra.

Consider

- *coproduct*: $\Gamma : VN(G) \to VN(G)\bar{\otimes}VN(G)$, $\lambda(s) \mapsto \lambda(s) \otimes \lambda(s)$,
- *antipode*: $S : VN(G) \mapsto VN(G)$, $\lambda(s) \mapsto \lambda(s^{-1})$.

If $G$ is abelian, then $VN(G) \simeq L^{\infty}(\hat{G})$, the coproduct is $\Gamma(f)(s,t) = f(st) \in L^{\infty}(G)\bar{\otimes}L^{\infty}(G)$, antipode $S(f)(s) = f(s^{-1})$.

There is a *fundamental unitary* $W \in VN(G)\bar{\otimes}B(L^2(G))$ that implements the coproduct:

$$\Gamma(X) = W^*(1 \otimes X)W, X \in VN(G).$$

$(W\xi)(s,t) = \xi(ts,t)$, $\xi \in L^2(G) \otimes L^2(G)$.

We have: $G \simeq \lambda(G) = \{X \in VN(G) \setminus \{0\} : \Gamma(X) = X \otimes X\}$.

Recall that if $\mathcal{M}$ is a von Neumann algebra on $\mathcal{H}$, then a closed densely defined unbounded operator $T$ on $\mathcal{H}$ is *affiliated with $\mathcal{M}$* if $U^*TU = T$ for every unitary $U \in \mathcal{M}'$. We write $\overline{\mathcal{M}}$ for the set of all such elements.
Obs! If $T = U|T|$ is the polar decomposition of $T$, then $T \in \overline{\mathcal{M}}$ iff $U \in \mathcal{M}$ and $E_{|T|}(\Delta) \in \mathcal{M}$, $\Delta \in \mathfrak{B}(\mathbb{R})$, where $E_{|T|}(\cdot)$ is the spectral measure of positive operator $|T|$.

As $\Gamma(X) = W^*(1 \otimes X)W$, $X \in VN(G)$, we can extend $\Gamma$ to $\overline{VN(G)}$ to get a map $\Gamma : \overline{VN(G)} \to \overline{VN(G)\bar{\otimes}VN(G)}$.

## Definition

An *abstract ($\lambda$-) complexification* $G_{\mathbb{C},\lambda}$ of $G$ is defined as the set

$$G_{\mathbb{C},\lambda} = \{X \in \overline{VN(G)} \setminus \{0\} : \Gamma(X) = X \otimes X\}.$$

If $G$ is compact, $G_{\mathbb{C},\lambda}$ coincides with McKennon, Cartwright-McMullen abstract complexification $G_{\mathbb{C}}$.

Let

$$\Lambda = \{\alpha \in \overline{VN(G)} : \Gamma(\alpha) = \alpha \otimes 1 + 1 \otimes \alpha, \alpha^* = -\alpha\}.$$

As in the compact case we have

## Theorem (O.Giselsson-T.)

- $\alpha \in \Lambda \mapsto \exp(i\alpha)$ *is a bijection onto* $G_{\mathbb{C},\lambda} \cap \overline{VN(G)}^+$
- $G_{\mathbb{C},\lambda} = \{\lambda(s)\exp(i\alpha) : \alpha \in \Lambda, s \in G\}$
- *If* $G$ *is a connected Lie group with Lie algebra* $\mathfrak{g}$ *then* $\Lambda = \{\partial\lambda(X) : X \in \mathfrak{g}\}$ *and* $G_{\mathbb{C},\lambda} = \{\lambda(s)\exp(i\partial\lambda(X)) : X \in \mathfrak{g}, s \in G\}$, *where* $\partial\lambda(X) = \overline{d\lambda(X)}$ *and* $d\lambda(X)\xi = \frac{d}{dt}\lambda(\exp(tX))\xi|_{t=0}$, $\xi \in \mathcal{D}^\infty(\lambda)$.

## Questions

Is $G_{\mathbb{C},\lambda}$ always a group?Is there a good locally compact structure on $G_{\mathbb{C},\lambda}$? If $G$ is a connected Lie group what is a connection to the universal complexification of $G$?

## FOURIER ALGEBRA OF LOCALLY COMPACT GROUP

- The *Fourier algebra* $A(G)$ of a locally compact group $G$ is
  1. $A(G) = VN(G)_*$ with multiplication $u \cdot v = \Gamma_*(u \otimes v)$, where $\Gamma$ is the coproduct OR
  2. $A(G) = \{f * \check{g} : f, g \in L^2(G)\} \subset C_0(G)$, where $\check{g}(x) = g(x^{-1})$, with pointwise multiplication.
- $A(G)$ is a (non-closed) subalgebra of $C_0(G)$ which is a commutative Banach algebra with norm $\|u\|_{A(G)} = \inf_{u=f*\check{g}} \|f\|_2\|g\|_2$.
- the duality with $VN(G)$ is given by $\langle T, \bar{f} * \check{g}\rangle = \langle Tg, f\rangle$, in particular, $\langle \lambda(x), u\rangle = u(x)$, $x \in G$.
- (Eymard, 64'). We have a homeomorphism

$$\mathrm{Spec}A(G) \simeq G, \lambda(x) \mapsto x,$$

where $\mathrm{Spec}A(G)$ is the *Gelfand spectrum*, i.e. all (bounded) non-zero multiplicative linear maps from $A(G)$ into $\mathbb{C}$.

The homeomorphism can be seen as follows: $X \in VN(G) \setminus \{0\}$ is a character of $A(G)$ iff for any $u, v \in A(G)$,

$$\begin{aligned} \langle X \otimes X, u \otimes v \rangle &= \langle X, u \rangle \langle X, v \rangle = \langle X, uv \rangle \\ &= \langle X, \Gamma_*(u \otimes v) \rangle = \langle \Gamma(X), u \otimes v \rangle, \end{aligned}$$

i.e.

$$\Gamma(X) = X \otimes X,$$

and hence $X \in \lambda(G)$.

- If $G$ is abelian, the Fourier transform gives $A(G) \overset{\mathcal{F}_G}{\simeq} L^1(\hat{G})$.
- If $G$ is compact, $\mathrm{Trig}(G) \subset A(G)$ and

$$A(G) \simeq \{u : G \to \mathbb{C} : \|u\|_{A(G)} = \sum_{\pi \in \hat{G}} \|\hat{u}(\pi)\|_1 d_\pi < \infty\},$$

  where $\mathcal{F}_G(u)(\pi) = \hat{u}(\pi) = \int_G u(s)\pi(s^{-1})ds$, $d_\pi = \dim H_\pi$, $\|\cdot\|_1$ is the trace norm.

## WEIGHTS AND WEIGHTED FOURIER ALGEBRAS

A scalar weight function on $G$ is a measurable function $w : G \to \mathbb{R}^+$ such that

$$w(st) \leq w(s)w(t) \text{ for all } s, t \in G.$$

We will also assume that it is bounded bellow, i.e. $\omega := w^{-1} \in L^\infty(G)$ and $\omega \otimes \omega \leq \Gamma(\omega)$, e.g. $G = \mathbb{R}$ or $\mathbb{Z}$, $w(x) = \beta^{|x|}(1 + |x|)^s$.

- If $G$ is abelian and $w$ is a weight on $\hat{G}$, we consider $L^1(\hat{G}, w)$ which is a Banach algebra (w.r.t. convolution) and weighted norm and let

$$A(G, w) := \mathcal{F}_{\hat{G}} L^1(\hat{G}, w).$$

- If $G$ is compact we define a weight function on the dual $\hat{G}$ of $G$ as $w : \hat{G} \to \mathbb{R}^+$ such that

$$w(\sigma) \leq w(\pi)w(\rho) \text{ for any } \sigma \subset \pi \otimes \rho.$$

Letting $\omega := \oplus_{\pi \in \hat{G}} w^{-1}(\pi) I_\pi \in VN(G)$, we obtain $\omega \otimes \omega \leq \Gamma(\omega)$. We define

$$A(G, \omega) := \{u : G \to \mathbb{C} : \|u\|_\omega = \sum_{\pi \in \hat{G}} \|\hat{u}(\pi)\|_1 w(\pi) d_\pi < \infty\}, \} = \omega \cdot A(G),$$

where $\omega \cdot u \in A(G)$ is given by $\langle T, \omega \cdot u\rangle = \langle T\omega, u\rangle$.

## Definition (general case)

A *weight inverse on the dual of $G$* is $\omega \in VN(G)$ such that

1. $\omega\omega^* \otimes \omega\omega^* \leq \Gamma(\omega\omega^*)$ ($\Leftrightarrow \omega \otimes \omega = \Gamma(\omega)\Omega$ for a unique $\Omega \in VN(G)$ with $\|\Omega\| \leq 1$)
2. $\ker \omega = \ker \omega^* = \{0\}$

We let $A(G,\omega) := \omega \cdot A(G) \subset A(G)$. It is a subalgebra of $A(G)$ as:

$$(\omega \cdot u)(\omega \cdot v) = \Gamma_*((\omega \otimes \omega) \cdot (u \otimes v)) = \omega \cdot (\Gamma_*(\Omega(u \otimes v))).$$

It is a commutative Banach algebra, called a *Beurling-Fourier algebra of $G$*, with respect to $\|\omega \cdot u\|_{A(G,\omega)} = \|u\|_{A(G)}$ with the dual $A(G,\omega)^* = VN(G)$ and duality: $\langle T, \omega \cdot u\rangle_\omega := \langle T, u\rangle$, $T \in VN(G)$, $u \in A(G)$.

## Question

What is the spectrum of $A(G,\omega)$? Its relation to the complexification?

*Beurling-Fourier algebras were introduced simultaneously in 2012 by H.H. Lee, E.Samei (JFA, 2012) and J.Ludwig, N.Spronk, T. (JFA, 2012)*

# $\mathrm{S}\textsc{pec}(A(G,\omega))$ AND COMPLEXIFICATION: COMPACT $G$

If $G$ is a compact group then $\mathrm{Trig}(G) \subset A(G,\omega)$ as a dense subalgebra for any weight inverse $\omega$.

If $T = (T_\pi)_{\pi\in\hat{G}} \in \mathrm{Spec}(A(G,\omega)) \subset VN(G) \simeq \oplus^{\ell^\infty}_{\pi\in\hat{G}}\mathcal{L}(H_\pi)$, then

$$\langle T, uv\rangle_\omega = \langle T,u\rangle_\omega \langle T,v\rangle_\omega, u,v \in \mathrm{Trig}(G).$$

As $\langle T,u\rangle_\omega = \langle T, \omega\cdot(\omega^{-1}\cdot u)\rangle_\omega = \langle T, \omega^{-1}\cdot u\rangle = \sum_{\pi\in\hat{G}} d_\pi \mathrm{Tr}(T_\pi\omega^{-1}(\pi)\hat{u}(\pi))$, $T\omega^{-1} \in G_{\mathbb{C}}$. Identifying $VN(G)$ with $VN(G,\omega^{-1}) := \{X\omega^{-1} : X \in VN(G)\}$, $X \mapsto X\omega^{-1}$ we have $\mathrm{Spec}(A(G,\omega)) \subset G_{\mathbb{C}}$

## Examples

1. $G = \mathbb{T}$, and $\omega_\beta : \mathbb{Z} \to \mathbb{R}^+, \omega_\beta(n) = \beta^{-|n|}$, $\beta \geq 1$ then

$$\mathbb{T}_{\mathbb{C}} \simeq \{(x_n)_n : \mathbb{Z} \to \mathbb{C}^* : x_{n+m} = x_n x_m \forall n,m \in \mathbb{Z}\} = \{(c^n)_n : c \in \mathbb{C}^*\}$$

and $\mathrm{Spec}(A(\mathbb{T},\omega)) \simeq \{c \in \mathbb{C} : \frac{1}{\beta} \leq |c| \leq \beta\} \subset \mathbb{C}^* = \mathbb{T}_{\mathbb{C}}$ as we need $(c^n\beta^{-|n|})_n \in \ell^\infty$.

2. $G = \mathbb{T}$ and $\omega(n) = (1+|n|)^{-\alpha}$, $\alpha \geq 0$ then $(c^n\omega(n)) \in \ell^\infty$ iff $c \in \mathbb{T}$ and $\mathrm{Spec}(A(\mathbb{T},\omega)) \simeq \mathbb{T}$.

# Exmaples, cont.

## Examples

3. $G = SU(2)$, $\hat{G} = \{\pi_n : n \in \mathbb{N} \cup \{0\}\}$, where $\pi_1 = \mathrm{id}$ and

$$\pi_n \otimes \pi_m \simeq \pi_{n+m} \oplus \pi_{n+m-2} \oplus \ldots \oplus \pi_{|n-m|}.$$

We have $\pi_1$ generates $\hat{G}$ in the sense that $\pi \in \hat{G} \Rightarrow \pi \subset \pi_1^{\otimes n}$ for some $n$.
$\omega : \hat{G} \to \mathbb{R}^+$ is a weight inverse iff $\tilde{\omega}(n) = \omega(\pi_n)$ is a weight inverse on $\mathbb{N} \cup \{0\}$ (e.g. $\tilde{\omega}(n) = \alpha^{-n}$, $\alpha > 1$) $\mathfrak{g} \simeq su(2)$, $\mathfrak{g}_{\mathbb{C}} \simeq sl(2, \mathbb{C})$ and $G_{\mathbb{C}} \simeq SL(2, \mathbb{C})$.
Let $\rho_\omega = \lim_{n\to\infty} \tilde{\omega}(n)^{1/n}$, $\Lambda \in SL(2, \mathbb{C})^+$, $(\pi_1)_{\mathbb{C}}(\Lambda) = \Lambda \simeq \begin{bmatrix} \lambda & \\ & \lambda^{-1} \end{bmatrix}$,
$\Lambda^{\otimes n} = (\pi_1)_{\mathbb{C}}^{\otimes n}(\Lambda) \Rightarrow \|(\pi_n)_{\mathbb{C}}(\Lambda)\| = \max(\lambda^n, \lambda^{-n})$.
Hence

$$\sup_{n\in\mathbb{N}} \|(\pi_n)_{\mathbb{C}}(\Lambda)\| \tilde{\omega}(n) = \sup_{n\in\mathbb{N}} \max(\lambda^n, \lambda^{-n}) \tilde{\omega}(n) < \infty \Leftrightarrow \max(\lambda, \lambda^{-1}) \rho_\omega \leq 1$$

and

$$\mathrm{Spec}(A(G,\omega)) \simeq \{x \in SL(2,\mathbb{C}) : \sigma(|x|) = \{\lambda, \lambda^{-1}\}, \rho_\omega \leq \lambda \leq 1/\rho_\omega\}$$

# $\mathrm{Spec}(A(G,\omega))$ and complexification: general $G$

Let $\omega \in VN(G)$ be a weight inverse and let $\Omega \in VN(G)\bar{\otimes}VN(G)$ be such that $\Gamma(\omega)\Omega = \omega \otimes \omega$ ($\|\Omega\| \leq 1$). Then $\Omega$ is a *2-cocycle*, i.e.

$$(\iota \otimes \Gamma)(\Omega)(I \otimes \Omega) = (\Gamma \otimes \iota)(\Omega)(\Omega \otimes I).$$

Identifying $\mathrm{Spec}(A(G,\omega))$ with a subset of $A(G,\omega)^* = VN(G)$, we have that

$$\mathrm{Spec}(A(G,\omega)) = \{\sigma \in VN(G) : \sigma \neq 0, \Gamma(\sigma)\Omega = \sigma \otimes \sigma\}.$$

In fact,

$$\begin{aligned}\langle \sigma \otimes \sigma, u \otimes v\rangle &= \langle \sigma, \omega \cdot u\rangle_\omega \langle \sigma, \omega \cdot v\rangle_\omega = \langle \sigma, (\omega \cdot u)(\omega \cdot v)\rangle_\omega \\ &= \langle \sigma, \omega \cdot (\Gamma_*(\Omega(u \otimes v)))\rangle_\omega = \langle \sigma, \Gamma_*(\Omega(u \otimes v))\rangle = \langle \Gamma(\sigma)\Omega, u \otimes v\rangle\end{aligned}$$

What are solutions to $\Gamma(\sigma)\Omega = \sigma \otimes \sigma$?

# SOLUTIONS TO $\Gamma(\sigma)\Omega = \sigma \otimes \sigma$?

*Guess*: $\sigma = T_\sigma \omega$ for a $T_\sigma \in G_{\mathbb{C},\lambda}$, since formally

$$\Gamma(T_\sigma)(\omega\xi \otimes \omega\eta) = \Gamma(T_\sigma)\Gamma(\omega)\Omega(\xi \otimes \eta) = \Gamma(\sigma)\Omega(\xi \otimes \eta) = \sigma\xi \otimes \sigma\eta$$

On the other hand,

$$(T_\sigma \otimes T_\sigma)(\omega\xi \otimes \omega\eta) = \sigma\xi \otimes \sigma\eta$$

so that $\Gamma(T_\sigma) = T_\sigma \otimes T_\sigma$ and $T_\sigma \in G_{\mathbb{C},\lambda}$. Hence if we can show that the operator $T_\sigma : \omega\xi \mapsto \sigma\xi$ is closable, then the closure $T_\sigma \in G_{\mathbb{C},\lambda}$.

## CLOSABILITY OF $T_\sigma : \omega\xi \mapsto \sigma\xi$

Let $S$ be the antipode on $VN(G)$, an anti-homomorphism given by $S(\lambda(s)) = \lambda(s^{-1})$, $s \in G$.

**Theorem [O.Giselsson-T.]**

If $\sigma \in \mathrm{Spec}A(G, \omega)$ then $T_\sigma : \omega\xi \mapsto \sigma\xi$ is closable and hence $T_\sigma \in G_{\mathbb{C},\lambda}$ if the following hold:

1. $\ker \sigma^* = \{0\}$;
2. $S(\sigma)\sigma = S(\omega)\omega$.

If $\sigma = T\omega$ with $T \in G_{\mathbb{C},\lambda}$ then formally $S(T) = T^{-1}$ and as $S$ is an anti-homomorphism

$$S(\sigma)\sigma = S(\omega)T^{-1}T\omega = S(\omega)\omega.$$

If (1) and (2) hold then

$$\omega\xi_n \to 0 \text{ and } \sigma\xi_n \to \eta \Rightarrow S(\sigma)\sigma\xi_n \to S(\sigma)\eta = 0 = \lim_n S(\omega)\omega\xi_n$$

and $\eta = 0$ as $\ker S(\sigma) = \{0\}$.

**Proposition [O.Giselsson-T.]**

If $\ker \Omega^* = \{0\}$ then $\ker \sigma^* = \{0\}$ for all $\sigma \in \mathrm{Spec}(A(G,\omega))$.

Aim: To prove $S(\sigma)\sigma = S(\omega)\omega$ for any $\sigma \in \mathrm{Spec}(A(G,\omega))$.

Let $W$ be the fundamental unitary that implements $\Gamma$, i.e. $\Gamma(X) = W^*(I \otimes X)W$. Then

$$S((\iota \otimes g)(W)) = \iota \otimes g(W^*) \text{ for all } g \in B(L^2(G))_*.$$

Consider $M = (S \otimes \iota)(W\Omega)W\Omega$. From $\Gamma(\sigma)\Omega = \sigma \otimes \sigma$ we have $(I \otimes \sigma)W\Omega = W(\sigma \otimes \sigma)$ and $(I \otimes \sigma)(S \otimes \iota)(W\Omega) = (S(\sigma) \otimes I)W^*(I \otimes \sigma)$ and

$$(I \otimes \sigma)M = (S(\sigma) \otimes I)W^*(I \otimes \sigma)W\Omega = S(\sigma)\sigma \otimes \sigma = (I \otimes \sigma)(S(\sigma)\sigma \otimes I)$$

If $\sigma = \omega$ ($\ker \omega = \{0\}$) it gives $M = S(\omega)\omega \otimes I$ and $(I \otimes \sigma)M = S(\omega)\omega \otimes \sigma$.

Obs! Calculations are only formal, as $S$ is not completely bounded in general.

**Proposition [O.Giselsson-T.]**

If $\sigma^*(H) \cap \omega^*(H) \neq \{0\}$ then (2) holds, i.e. $S(\sigma)\sigma = S(\omega)\omega$.

# RESULTS

## Proposition [O.Giselsson-T.]

If there exists $K \subset H$ such that $VN(G)(K) \subset K$ and $\omega|_K$ is invertible then (2) holds for any $\sigma \in \mathrm{Spec}A(G,\omega)$, e.g. when

- $G$ is compact;
- $G$ is Moore;
- $G$ is a seprable type $I$-group $\omega = \int_{\xi \in \hat{G}}^{\oplus} \omega_\xi d\mu(\xi)$ and $\omega_\xi$ is invertible almost everywhere.

## Extensions of weights from subgroups

Let $H$ be a closed subgroup of $G$ and let $\iota : VN(H) \to VN(G)$, $\lambda_H(g) \mapsto \lambda_G(g)$. If $\omega_H$ is a weight inverse on the dual of $H$ then $\omega_G = \iota(\omega_H)$ is a weight inverse on the dual of $G$.

## Theorem [O.Giselsson-T.]

If $H$ is an abelian subgroup of $G$ and $\omega_H$ is a weight inverse then (1) and (2) holds for any $\sigma \in \mathrm{Spec}(A(G,\iota(\omega_H)))$ and $\mathrm{Spec}(A(G,\iota(\omega_H))) \subset G_{\mathbb{C},\lambda}$.

## Theorem [O.Giselsson-T., GLLST]

Let $G$ be one of the following groups:

1. a connected simply connected nilpotent Lie group;
2. the reduced Heisenberg groups;
3. the Euclidean motion group $E(2)$.
4. simply connected cover $\tilde{E}(2)$ of $E(2)$

and let $H$ be an abelian connected closed subgroup of $G$. Suppose $\omega_H$ is a weight inverse on the dual of $H$ and let $\omega_G = \iota(\omega_H)$. Then (with identification $VN(G) \simeq VN(G, \omega_G^{-1})$)

$$\begin{aligned} \mathrm{Spec}(A(G,\omega_G)) \;\; &\simeq \;\; \{\lambda_G(s)\exp(i\partial\lambda_G(X)) : s \in G, X \in \mathfrak{h}, \\ &\qquad \exp(i\partial\lambda_H(X)) \in \mathrm{Spec}(A(H,\omega_H))\}. \end{aligned}$$

# IDEA OF THE PROOF

**Step 1**: As $\omega_G$ is is induced from a weight on the dual of abelian subgroup,

$$\mathrm{Spec}(A(G,\omega_G)) \subset G_{\mathbb{C},\lambda} = \{\lambda_G(s)\exp(i\partial\lambda_G(X)) : s \in G, X \in \mathfrak{g}\}.$$

**Step 2**: If $\mathfrak{h}$ is the Lie algebra of $H$ with the basis $X_1, \ldots, X_m$ and $\omega_H = \omega(i\partial\lambda_H(X_1), \ldots, i\partial\lambda_H(X_m))$ for a weight inverse function $\omega : \mathbb{R}^m \to \mathbb{R}^+$ then $\omega_G = \omega(i\partial\lambda_G(X_1), \ldots, i\partial\lambda_G(X_m))$. We show that

$$X \in \mathfrak{g} \setminus \mathfrak{h} \Rightarrow \exp(i\partial\lambda_G(X))\omega_G \text{ is unbounded.}$$

**Step 3**: Show that for $X \in \mathfrak{h}$,
$\exp(i\partial\lambda_G(X)) = \iota(\exp(i\partial\lambda_H(X))) \in \mathrm{Spec}(A(G,\iota(\omega_H)))$ iff $\exp(i\partial\lambda_H(X)) \in \mathrm{Spec}(A(H,\omega_H))$.

# EXAMPLE: HEISENBERG GROUP.

- $\mathbb{H} = \left\{ (y,z,x) = \begin{bmatrix} 1 & x & z \\ & 1 & y \\ & & 1 \end{bmatrix} : x, y, z \in \mathbb{R} \right\}$
- $H = H_{Y,Z} = \{(y,z,0) \in \mathbb{H} : y, z \in \mathbb{R}\}$-abelian subgroup
- We have the universal complexification

$$\mathbb{H}_{\mathbb{C}} = \left\{ (y,z,x) = \begin{bmatrix} 1 & x & z \\ & 1 & y \\ & & 1 \end{bmatrix} : x, y, z \in \mathbb{C} \right\}$$

  and Cartan type decomposition $\mathbb{H}_{\mathbb{C}} \simeq \mathbb{H} \cdot \exp(i\mathfrak{h})$, where $\mathfrak{h}$ is the Lie algebra of $\mathbb{H}$.

Let $\omega$ be a weight inverse on $\mathbb{R}^2$. Then $\omega_H = (\mathcal{F}^H)^{-1} M_\omega \mathcal{F}^H \in VN(H_{Y,Z})$ is a weight inverse and the condition $\exp(i\partial\lambda_H(X)) \in \mathrm{Spec}(A(H_{Y,Z}, \omega_H))$ for $X = (y,z,0) \in \mathfrak{h}$ is equivalent to

$$e^{ay} e^{bz} \omega(a,b) \in L^\infty(\mathbb{R}^2).$$

# HEISENBERG GROUP, CONT.

When $\omega(a,b) = \beta_1^{-|a|}\beta_2^{-|b|} \in L^\infty(\mathbb{R}^2)$, $(a,b) \in \mathbb{R}^2$ for some $\beta_1, \beta_2 \geq 1$ we get

$$\begin{aligned} \mathrm{Spec}(A(\mathbb{H}, \iota(\omega_H))) &\simeq \{g \cdot (iy, iz, 0) \in \mathbb{H}_\mathbb{C} \simeq \mathbb{C}^3 : g \in \mathbb{H}, \\ &\quad y, z \in \mathbb{R}, |y| \leq \ln \beta_1, |z[\leq \ln \beta_2\}. \end{aligned}$$

When $\omega(a,b) = (1 + \|(a,b)\|)^{-\alpha}$, $\alpha > 0$ then $\mathrm{Spec}(A(\mathbb{H}, \iota(\omega_H))) \simeq \mathbb{H}$.

Thank You!

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