The parabolic algebra revisited

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A (concrete) operator algebra is a subalgebra of $B(H)$, for some Hilbert space H.

Algebra :

An operator algebra is called selfajoint, if it is closed under the involution map

$$T \mapsto T^*, \text{ where } \langle Tx, y\rangle = \langle x, T^* y\rangle, \ \forall x, y \in H.$$

Topologies :

- Norm Topology : $T_n \xrightarrow{\|\cdot\|} T$, when $\sup\{\|T_n x - Tx\| : x \in B_H\} \xrightarrow{n} 0$
- SOT Topology : $T_i \xrightarrow{SOT} T$, when $T_i x \xrightarrow{i} Tx$, $\forall x \in H$.
- WOT Topology : $T_i \xrightarrow{WOT} T$, when $\langle T_i x, y\rangle \xrightarrow{i} \langle Tx, y\rangle$, $\forall x, y \in H$.
- w*-topology : $T_n \xrightarrow{w^*} T$, when $\sum_{n=1}^{\infty} \lambda_n \langle (T_n - T)x_n, y_n\rangle \to 0$, for all $(\lambda_n) \in \ell^1$, $\|x_n\| = \|y_n\| = 1$.

Operator algebras	selfadjoint	non-selfadjoint
norm topology	$C(\mathbb{T})$	$A(\mathbb{T})$
w*-topology	$L^\infty(\mathbb{T})$	$H^\infty(\mathbb{T})$

Write $F = \text{alg}\{e^{inx} : n \in \mathbb{N}\}$ and $F_+ = \text{alg}\{e^{inx} : n \geq 0\}$. Then

Operator algebras	selfadjoint	non-selfadjoint
norm topology	$\overline{F}^{\|\cdot\|}$	$\overline{F_+}^{\|\cdot\|}$
w*-topology	$\overline{F}^{w^*}$	$\overline{F_+}^{w^*}$

Let H be a Hilbert space and $\mathcal{S} \subseteq \mathbf{B}(H)$. We define its **commutant** to be the set :

$$\mathcal{S}' = \{T \in B(H) : TS = ST, \forall S \in \mathcal{S}\}$$

The set $\mathcal{S}'$ is a unital algebra and it is SOT-closed.

Let H be a Hilbert space. A **von Neumann algebra** $\mathcal{M}$ acting on H is a selfadjoint subset of $\mathbf{B}(H)$, that satisfies the property $\mathcal{M} = \mathcal{M}''$.

Bicommutant Theorem

If $\mathcal{A} \subseteq \mathbf{B}(H)$ is a unital selfadjoint algebra, then

$$\mathcal{A}'' = \overline{\mathcal{A}}^{w*}.$$

In fact , the following are equivalent :

(α) $\mathcal{A}$ is a von Neumann algebra

(β) $\mathcal{A}$ is w*-closed

Proposition

Let $\mathcal{M}$ be a von Neumann algebra acting on Hilbert space H.Then $\mathcal{M}$ is the norm closed linear span of its projections.
Furthermore, if H is separable, there is a countable set $\mathcal{E} \subseteq \mathcal{M}$ of projections, such that $\mathcal{E}'' = \mathcal{M}$.

Note: A C^*-algebra may have no non-trivial projections.

What happens with non-selfadjoint algebras?

Given a set of operators $\mathcal{A}$, the lattice of all invariant subspaces of $\mathcal{A}$ is denoted $Lat\mathcal{A}$.

Similarly, if $\mathcal{L}$ is a set of subspaces, then $Alg\mathcal{L}$ denotes the algebra of all operators leaving each element of $\mathcal{L}$ invariant.

Definition

An algebra $\mathcal{A}$ is **reflexive** if $\mathcal{A} = Alg\ Lat\mathcal{A}$.
A lattice $\mathcal{L}$ is **reflexive** if $\mathcal{L} = Lat\ Alg\mathcal{L}$.

Do we get an analogue of von Neumann theorem?

No. Take $\mathcal{A} = \left\{\begin{pmatrix} a & b \\ 0 & a \end{pmatrix} : \ a, b \in \mathbb{C}\right\}$ and $\mathcal{B} = \left\{\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} : \ a, b, c \in \mathbb{C}\right\}$.
Then $\mathcal{A}$ is w*-closed, but $Alg\ Lat\mathcal{A} = \mathcal{B}$.

Examples of reflexive algebras:

- H^∞ as multiplication algebra on $L^2(\mathbb{R})$.
- Nest algebras.

A **nest** $\mathcal{N}$ is a totally ordered lattice of closed subspaces of a Hilbert space H containing $\{0\}$ and H. The **nest algebra** is $Alg\mathcal{N}$.

Theorem (Erdos Density theorem)

The finite rank contractions in a nest algebra $\mathcal{A}$ are dense in the unit ball of $\mathcal{A}$ in the SOT-topology.

Examples:

- Let $\{e_i \,|\, i = 1, \ldots, n\}$ be a basis in $\mathbb{C}^n$ and $N_k = span\{e_1, \ldots, e_k\}$. Then the set $\mathcal{N} = \{N_k : k \in \{1, \ldots, n\}\} \cup \{0\}$ is a nest. The nest algebra $Alg\mathcal{N}$ consists of all the upper triangular matrices with respect to the basis.
- Let $H = L^2(\mathbb{R})$. The **Volterra nest** is the family $\mathcal{N}_v = \{N_t : t \in \mathbb{R}\} \cup \{\{0\}, L^2(\mathbb{R})\}$, where

$$N_t = \{f \in L^2(\mathbb{R}) \,\big|\, f(x) = 0 \text{ a.e. on } (-\infty, t]\}.$$

- The **analytic nest** :

$$\mathcal{N}_a = \mathcal{F}^* \mathcal{N}_v = \{e^{i\lambda x} H^2(\mathbb{R}) : \lambda \in \mathbb{R}\} \cup \{\{0\}, L^2(\mathbb{R})\} \ \text{(Paley - Wiener)}$$

Our favorite operators in $L^2(\mathbb{R})$.

- $(M_{\lambda}f)(x) = e^{i\lambda x}f(x)$, multiplication operator
- $(D_{\mu}f)(x) = f(x-\mu)$, translation operator
- $(V_t f)(x) = e^{t/2}f(e^t x)$, dilation operator
- $(M_{\phi_s}f)(x) = e^{-isx^2/2}f(x)$

Note that $\mathcal{A}_v = \overline{alg\{M_{\lambda}D_{\mu} | \lambda \in \mathbb{R}, \mu \geq 0\}}^{w^*}$.

Define

$$\mathcal{A}_p = \overline{alg\{M_\lambda D_\mu | \lambda, \mu \geq 0\}}^{w^*}$$

Theorem (Katavolos - Power)

Define $\mathcal{A}_{FB} = Alg(\mathcal{N}_a \cup \mathcal{N}_v)$. The non-selfadjoint algebra $\mathcal{A}_{FB}$ is called the **Fourier Binest Algebra**. Then

$$\mathcal{A}_{FB} = \mathcal{A}_p.$$

Properties:

- $\mathcal{A}_{FB}$ is a reflexive algebra
- $\mathcal{A}_{FB}$ contains no non-zero finite rank operators.
- $\mathcal{A}_{FB}$ contains no non-trivial selfadjoint operators, i.e. $\mathcal{A}_{FB} \cap \mathcal{A}_{FB}^* = \mathbb{C}I$.
- Its automorphism group is generated by $\{M_\lambda, D_\mu, V_t \ : \ \lambda, \mu, t \in \mathbb{R}\}$.

Sketch of proof:

Proposition

Given $k \in L^2(\mathbb{R}^2)$, define $\Theta(k) : (x,t) \mapsto k(x, x-t)$. Then

$$\mathcal{A}_{FB} \cap C_2 = \{Intk | \Theta(k) \in H^2(\mathbb{R}) \otimes L^2(\mathbb{R}_+)\}$$

Take $h \in H^2(\mathbb{R})$, $\phi \in L^2(\mathbb{R}_+)$. Then $Int(\Theta^{-1}(h \otimes \phi))$ lies in $\mathcal{A}_p$. Thus

Proposition

$$\mathcal{A}_{FB} \cap C_2 = \mathcal{A}_p \cap C_2$$

The theorem follows from the fact that C_2 is an ideal in the $B(L^2(\mathbb{R}))$, and

Proposition

$\mathcal{A}_p$ has a bounded approximate identity of Hilbert Schmidt operators in SOT-topology.

Theorem (K.)

In L^p spaces, $\mathcal{A}^p_{FB} = \mathcal{A}^p_p$, $(1 < p < \infty)$.

Theorem (Katavolos - Power)

The Fourier binest is not reflexive.

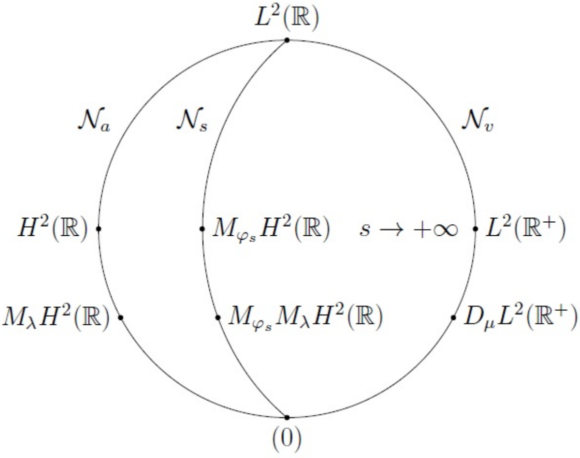

Figure: $Lat\mathcal{A}_p$

Figure: $Lat\mathcal{A}_p$ (Copyright : AK)

Let $\mathcal{L}$ be a reflexive subspace lattice and $\mathcal{B}$ be a maximal abelian subalgebra of Alg $\mathcal{L}$. Then $\mathcal{L}$ is said to be **synthetic relative to** $\mathcal{B}$ if Alg $\mathcal{L}$ is the only w*-closed algebra $\mathcal{A}$ with $Lat\mathcal{A} = \mathcal{L}$ and $\mathcal{B} \subseteq \mathcal{A}$.

Example: The Volterra nest is synthetic relative to the multiplicative masa $\mathcal{M}_{L^\infty}$ on $L^2(\mathbb{R})$.

Theorem (K.-Power)

The lattice $Lat\mathcal{A}_p$ fails to be synthetic relative to the maximal abelian algebra $\mathcal{M}_{H^\infty}$.

Sketch of proof:

Take $\mathcal{A}_0 = w^* - alg\big(\{M_\lambda, M_\lambda D_\mu \ : \ \lambda \geq 0, \mu \geq 2\} \cup \{M_\lambda D_\mu \ : \ \lambda \geq 2, \mu \geq 0\}\big)$.

Figure: Generators of $\mathcal{A}_0$

- Show by cocycle argument (as in $Lat\mathcal{A}_p$) that $Lat\mathcal{A}_0 = Lat\mathcal{A}_p$
- Find w^*-continuous functional to separate the two algebras. Check functionals of the form:

$$\omega(T) = \sum_{n \in \mathbb{N}} a_n \langle T M_{\phi_1} F^{-1} f_n, M_{\phi_1} F^{-1} g_n \rangle,$$

where $\{a_n\} \in \ell^1, \ f_n, g_n \in L^2[0,2]$ and $||f_n||_2 = ||g_n||_2 = 1$.

Question: Is the w*-closed algebra generated by the multiplication and translation operators still reflexive, if we restrict on the Hardy space?

Theorem (K.-Power)

Let K be a non-trivial subspace of $Lat\mathcal{A}_p$. The w*-closure of the restriction of the parabolic algebra on K is reflexive and unitarily equivalent to the Volterra nest algebra.

Idea: $\overline{P_{L^2(\mathbb{R}_+)}(H^\infty(\mathbb{R}))}^{w^*} = L^\infty(\mathbb{R}_+)$.

By the term **quasicompact algebra**, we mean an algebra of the type

$$Q\mathcal{A} = (\mathcal{A} + K(H)) \cap (\mathcal{A}^* + K(H)),$$

Theorem (K.-Power)

The quasicompact algebra of $\mathcal{A}_p$ strictly larger than the algebra $\mathcal{A}_p \cap \mathcal{A}_p^* + K = \mathbb{C}I + K$.

Lemma

The restriction of the triangular truncation operator $\mathcal{P}_v\big|_{\mathcal{A}_p + \mathcal{A}_p^*}$ is unbounded.

Take $T_n \in A_p \cap K$ such that $||T_n|| = 1$ and $||T_n - T_n^*|| \to 0$.
Define $A_n = D_{t_n} T_n D_{t_n}^*$ and $X_n = A_n - A_n^*$, where t_n is large enough.
Set $A = SOT - \sum A_n$ and $X = A - A^*$. Then $X = || \cdot || - \sum X_n \in K(H)$.
Show that $A \notin \mathbb{C}I + K$.

Questions:

- Is the parabolic algebra an integral domain ?
- What about the w*-closed ideals of $\mathcal{A}_p$?
- Could $\mathcal{A}_p^p, \mathcal{A}_p^q$ be isomorphic for different p, q?
- Is $(\mathcal{A}_v + K) \cap (\mathcal{A}_a + K)$ strictly larger than the algebra $\mathcal{A}_v \cap \mathcal{A}_a + K = \mathcal{A}_p + K$?

Thank you!