Duality for crossed products of operator spaces and the approximation property of Haagerup-Kraus

Dimitrios Andreou

National and Kapodistrian University of Athens, Greece

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Spatial & Fubini tensor product

A dual operator space is a w*-closed subspace of $\mathcal{B}(H)$ for some Hilbert space H.

- For dual operator spaces $X \subseteq \mathcal{B}(H)$ and $Y \subseteq \mathcal{B}(K)$, the spatial tensor product of X and Y is the subspace of $\mathcal{B}(H \otimes K)$ defined by

$$X \overline{\otimes} Y = \overline{\operatorname{span}}^{w^*} \{x \otimes y : x \in X, y \in Y\},$$

where $(x \otimes y)(h \otimes k) = (xh) \otimes (yk)$, for $h \in H, k \in K$.

- The Fubini tensor product of X and Y is the space:

$$X \overline{\otimes}_{\mathcal{F}} Y = (X \overline{\otimes} \mathcal{B}(K)) \cap (\mathcal{B}(H) \overline{\otimes} Y) \simeq \left(X_* \widehat{\otimes} Y_*\right)^* .$$

- Obviously, $X \overline{\otimes} Y \subseteq X \overline{\otimes}_{\mathcal{F}} Y$ for all dual operator spaces X and Y.
- We say that Y has property S_σ if $X \overline{\otimes} Y = X \overline{\otimes}_{\mathcal{F}} Y$ for any dual operator space X.
- (Kraus) Every *injective* von Neumann algebra M (e.g. of type I) has property S_σ.
- (Effros-Ruan) If M and N are von Neumann algebras, then $M \overline{\otimes} N = M \overline{\otimes}_{\mathcal{F}} N$.

Comodules-basic definitions

A Hopf-von Neumann algebra (HvNa) is a pair (M, Δ), where M is a von Neumann algebra and $\Delta\colon M \to M\overline{\otimes}M$ is a comultiplication, that is a normal unital *-injection which is coassociative:

$$(\Delta \otimes \mathrm{id}_M) \circ \Delta = (\mathrm{id}_M \otimes \Delta) \circ \Delta$$

$$\begin{tikzcd} M \arrow[r, "\Delta"] \arrow[d, "\Delta"'] & M\overline{\otimes}M \arrow[d, "\Delta\otimes\mathrm{id}_M"] \\ M\overline{\otimes}M \arrow[r, "\mathrm{id}_M\otimes\Delta"'] & M\overline{\otimes}M\overline{\otimes}M \end{tikzcd}$$

An M-comodule is a pair (X, α), where X is a dual operator space and $\alpha\colon X \to X\overline{\otimes}_{\mathcal{F}}M$ is an M-action on X, i.e. a w*-continuous complete isometry which is coassociative over Δ:

$$(\alpha \otimes \mathrm{id}_M) \circ \alpha = (\mathrm{id}_X \otimes \Delta) \circ \alpha$$

$$\begin{tikzcd} X \arrow[r, "\alpha"] \arrow[d, "\alpha"'] & X\overline{\otimes}_{\mathcal{F}}M \arrow[d, "\alpha\otimes\mathrm{id}_M"] \\ X\overline{\otimes}_{\mathcal{F}}M \arrow[r, "\mathrm{id}_X\otimes\Delta"'] & X\overline{\otimes}_{\mathcal{F}}M\overline{\otimes}_{\mathcal{F}}M \end{tikzcd}$$

The fixed point space of X is the subspace $X^\alpha = \{x \in X : \alpha(x) = x \otimes 1_M\}$.
An M-subcomodule of X is w*-closed subspace $Y \subseteq X$ such that $\alpha(Y) \subseteq Y\overline{\otimes}_{\mathcal{F}}M$, i.e. $(Y, \alpha|_Y)$ is an M-comodule.

Comultiplications on $L^\infty(G)$ and $L(G)$

Let G be a locally compact group with left Haar measure and modular function Δ_G. Also, let $\lambda\colon G \to \mathcal{B}(L^2(G))$ be the left regular representation of G:

$$\lambda_s \xi(t) = \xi(s^{-1}t), \qquad \xi \in L^2(G).$$

- $L^\infty(G)$ is regarded as a von Neumann algebra acting on $L^2(G)$ by multiplication and it is a HvNa with comultiplication

$$\alpha_G\colon L^\infty(G) \to L^\infty(G \times G) \simeq L^\infty(G)\overline{\otimes}L^\infty(G),$$

$$\alpha_G(f)(s,t) = f(ts), \quad s,t \in G,\ f \in L^\infty(G).$$

- The left group von Neumann algebra $L(G) := \lambda(G)'' \subseteq \mathcal{B}(L^2(G))$ is also a HvNa with comultiplication

$$\delta_G\colon L(G) \to L(G)\overline{\otimes}L(G),$$

$$\delta_G(\lambda_s) = \lambda_s \otimes \lambda_s, \quad s \in G.$$

Why comodules?

- Any $L^\infty(G)$-action α on X corresponds to a strongly w*-continuous G-action $\gamma\colon G \to \mathrm{Aut}^{\mathrm{ci}}_{\mathrm{w}^*}(X)$ by completely isometric w*-continuous automorphisms,
$$\gamma_s = \alpha^{-1} \circ (\mathrm{id}_X \otimes \mathrm{Ad}\lambda_s) \circ \alpha,\ s \in G.$$
- In fact, this is a bijective correspondence since the $L^\infty(G)$-action α is uniquely determined by the G-action γ via
$$\langle \alpha(x), \omega \otimes h\rangle = \int_G \langle \gamma_s^{-1}(x), \omega\rangle h(s)\, \mathrm{d}s,\ x \in X,\ \omega \in X_*,\ h \in L^1(G),$$
using the duality $X \overline{\otimes} L^\infty(G) \simeq \left(X_* \widehat{\otimes} L^1(G)\right)^*$.
- $(\alpha \otimes \mathrm{id}) \circ \alpha = (\mathrm{id} \otimes \alpha_G) \circ \alpha \iff \gamma_s \circ \gamma_t = \gamma_{st}\ \forall s, t \in G.$
- $X^\alpha = \{x \in X : \gamma_s(x) = x\ \forall s \in G\}$.
- If G is abelian, then $L(G) \simeq L^\infty(\widehat{G})$ where $\widehat{G}$ is the dual group. In the non-abelian case, we have to replace dual group actions with $L(G)$-comodules since there is not dual group.

Spatial crossed product

For an $L^\infty(G)$-comodule (X, α) with $X \subseteq \mathcal{B}(H)$ the spatial crossed product of X by α is w*-closed $\mathbb{C}1_H \overline{\otimes} L(G)$-submodule of $X \overline{\otimes} \mathcal{B}(L^2(G))$ generated by $\alpha(X)$. That is the space

$$X \overline{\rtimes}_\alpha G = \overline{\text{span}}^{\text{w}^*} \{(1_H \otimes \lambda_s)\alpha(x) : \ s \in G, \ x \in X\}.$$

Remark: If α is trivial, i.e. $\alpha(x) = x \otimes 1$ for all $x \in X$, then $X \overline{\rtimes}_\alpha G = X \overline{\otimes} L(G)$.

Also, if X is a von Neumann algebra and α is a unital *-homomorphism induced by a G-action γ, then from the covariance relations

$$\alpha(\gamma_s(x)) = (1 \otimes \lambda_s)\alpha(x)(1 \otimes \lambda_s^{-1})$$

it follows that $X \overline{\rtimes}_\alpha G = (\alpha(X) \cup (\mathbb{C}1 \overline{\otimes} L(G)))''$, i.e. the usual von Neumann algebra crossed product.

Fubini crossed product

For an $L^\infty(G)$-comodule (X,α) we have an $L^\infty(G)$-action

$$\widetilde{\alpha}\colon X\overline{\otimes}\mathcal{B}(L^2(G)) \to X\overline{\otimes}\mathcal{B}(L^2(G))\overline{\otimes}L^\infty(G),$$

defined as the following composition:

$$\widetilde{\alpha} := (\mathrm{id}_X \otimes \mathrm{Ad}U_G^*) \circ (\mathrm{id}_X \otimes \sigma) \circ (\alpha \otimes \mathrm{id}_{\mathcal{B}(L^2(G))}),$$

$$\sigma(a\otimes b) = b\otimes a, \qquad a,b\in\mathcal{B}(L^2(G))$$

$$U_G f(s,t) = \Delta_G(t)^{1/2} f(st,t), \qquad f\in L^2(G\times G),\ s,t\in G.$$

$$\begin{tikzcd}
X\overline{\otimes}\mathcal{B}(L^2(G)) \arrow[r, "\alpha\otimes\mathrm{id}_{\mathcal{B}(L^2(G))}"] \arrow[drr, dashed, "\widetilde{\alpha}"'] & X\overline{\otimes}L^\infty(G)\overline{\otimes}\mathcal{B}(L^2(G)) \arrow[r, "\mathrm{id}_X\otimes\sigma"] & X\overline{\otimes}\mathcal{B}(L^2(G))\overline{\otimes}L^\infty(G) \arrow[d, "\mathrm{id}_X\otimes\mathrm{Ad}U_G^*"] \\
 & & X\overline{\otimes}\mathcal{B}(L^2(G))\overline{\otimes}L^\infty(G)
\end{tikzcd}$$

The Fubini crossed product of X by α is the fixed point space

$$X \rtimes_\alpha^{\mathcal{F}} G = \left(X\overline{\otimes}\mathcal{B}(L^2(G))\right)^{\widetilde{\alpha}} = \{y\in X\overline{\otimes}\mathcal{B}(L^2(G)) :\ \widetilde{\alpha}(y) = y\otimes 1\}.$$

Comments on the definition

1. Note that if the $L^\infty(G)$-action α on X corresponds to a G-action $\gamma\colon G \to \mathrm{Aut}^{\mathrm{ci}}_{\mathrm{w}^*}(X)$, then $\widetilde{\alpha}$ is the $L^\infty(G)$-action on $X\overline{\otimes}\mathcal{B}(L^2(G))$ which corresponds to the G-action $s \in G \mapsto \gamma_s \otimes \mathrm{Ad}\rho_s$, where ρ is the right regular representation
$$\rho_s\xi(t) = \Delta_G(s)^{1/2}\xi(ts), \qquad \xi \in L^2(G).$$
Thus we have
$$X \rtimes^{\mathcal{F}}_{\alpha} G = \{y \in X\overline{\otimes}\mathcal{B}(L^2(G)) : \ (\gamma_s \otimes \mathrm{Ad}\rho_s)(y) = y \ \forall s \in G\}.$$
So, if α is trivial, i.e. $\alpha(x) = x \otimes 1$ for all $x \in X$, then $\gamma_s = \mathrm{id}_X$ for all $s \in G$ and hence $X \rtimes^{\mathcal{F}}_{\alpha} G = X\overline{\otimes}_{\mathcal{F}}L(G)$.
2. Moreover, it is easy to verify the following
   - $(\mathbb{C}1_H\overline{\otimes}L(G))(X \rtimes^{\mathcal{F}}_{\alpha} G) \subseteq X \rtimes^{\mathcal{F}}_{\alpha} G$;
   - $\alpha(X) \subseteq X \rtimes^{\mathcal{F}}_{\alpha} G$

   and therefore
$$X\overline{\rtimes}_{\alpha}G \subseteq X \rtimes^{\mathcal{F}}_{\alpha} G.$$

$X\overline{\rtimes}_\alpha G = X \rtimes_\alpha^{\mathcal{F}} G$?

Question

Let G be a locally compact group and (X, α) an $L^\infty(G)$-comodule. Is it true that $X\overline{\rtimes}_\alpha G = X \rtimes_\alpha^{\mathcal{F}} G$?

Answer: Not always...

If $X\overline{\rtimes}_\alpha G = X \rtimes_\alpha^{\mathcal{F}} G$ for any $L^\infty(G)$-comodule (X, α), then $L(G)$ must have property S_σ.

This is because if α is the trivial action on X, then $X\overline{\rtimes}_\alpha G = X\overline{\otimes}L(G)$ and $X \rtimes_\alpha^{\mathcal{F}} G = X\overline{\otimes}_{\mathcal{F}}L(G)$.

Counterexample: $L(SL_3(\mathbb{Z}))$ does not have property S_σ.

However, the equality $X\overline{\rtimes}_\alpha G = X \rtimes_\alpha^{\mathcal{F}} G$ holds when X has some additional structure preserved by α, e.g.

1. (Digernes-Takesaki, 1975) If X is a von Neumann algebra and α is an $L^\infty(G)$-action on X which is a unital *-homomorphism, then $X \rtimes_\alpha^{\mathcal{F}} G = X\overline{\rtimes}_\alpha G$.
2. (Salmi-Skalski, 2015) If α is an $L^\infty(G)$-action on a (non-degenerately represented) W*-TRO X such that α is a (non-degenerate) TRO-morphism, then $X \rtimes_\alpha^{\mathcal{F}} G = X\overline{\rtimes}_\alpha G$.

Crossed products and the AP

Definition (Haagerup-Kraus)

A locally compact group G has the approximation property (AP) if there is a net $\{u_i\}$ in the Fourier algebra $A(G)$ such that $u_i \to \mathbf{1}$ in the $\sigma(M_{cb}A(G), Q(G))$-topology.

Theorem (Crann-Neufang, 2019)

Let G be a locally compact group and consider the following conditions:

(i) *G has the AP;*

(ii) *$X \overline{\rtimes}_\alpha G = X \rtimes_\alpha^{\mathcal{F}} G$ for any $L^\infty(G)$-comodule (X, α);*

(iii) *$L(G)$ has property S_σ.*

Then, (i) $\implies$ (ii) $\implies$ (iii).
If G is inner amenable (e.g. discrete), then (iii) $\implies$ (i).

We can prove at least that (ii) $\implies$ (i) for arbitrary locally compact groups using two main ingredients:

1. A duality theory for crossed products of operator spaces;
2. A characterization of the AP in terms of $L(G)$-comodules.

Crossed products of $L(G)$-comodules

For an $L(G)$-comodule (Y,δ) with $Y \subseteq \mathcal{B}(H)$ one can define the spatial crossed product of Y by δ

$$Y \overline{\ltimes}_\delta G = \overline{\text{span}}^{w^*} \{(1_H \otimes f)\,\delta(y) : \ f \in L^\infty(G),\ y \in Y\}$$

as well as the Fubini crossed product

$$Y \ltimes_\delta^{\mathcal{F}} G = \left(Y \overline{\otimes} \mathcal{B}(L^2(G))\right)^{\widetilde{\delta}},$$

where $\widetilde{\delta}\colon Y\overline{\otimes}\mathcal{B}(L^2(G)) \to (Y\overline{\otimes}\mathcal{B}(L^2(G)))\overline{\otimes}_{\mathcal{F}} L(G)$ is the $L(G)$-action

$$\widetilde{\delta} = (\mathrm{id}_Y \otimes \mathrm{Ad} W_G) \circ (\mathrm{id}_Y \otimes \sigma) \circ (\delta \otimes \mathrm{id}_{\mathcal{B}(L^2(G))})$$

and

$$W_G \xi(s,t) = \xi(s,st), \quad \xi \in L^2(G \times G),\ s,t \in G.$$

Dual actions

Let (X, α) be an $L^\infty(G)$-comodule and (Y, δ) be an $L(G)$-comodule with $X \subseteq \mathcal{B}(H)$ and $Y \subseteq \mathcal{B}(K)$

1. The map

$$\widehat{\alpha}(x) = (1_H \otimes W_G^*)(x \otimes 1_{L^2(G)})(1_H \otimes W_G), \quad x \in X \rtimes_\alpha^{\mathcal{F}} G,$$

   is an $L(G)$-action on $X \rtimes_\alpha^{\mathcal{F}} G$ called the dual of α.
   Moreover,

$$\widehat{\alpha}\,(X \overline{\rtimes}_\alpha G) \subseteq (X \overline{\rtimes}_\alpha G)\,\overline{\otimes}_{\mathcal{F}} L(G).$$

   That is $X \overline{\rtimes}_\alpha G$ is an $L(G)$-subcomodule of $(X \rtimes_\alpha^{\mathcal{F}} G, \widehat{\alpha})$.

2. The map

$$\widehat{\delta}(x) = (1_K \otimes U_G^*)(x \otimes 1_{L^2(G)})(1_K \otimes U_G), \quad x \in Y \ltimes_\delta^{\mathcal{F}} G$$

   is an $L^\infty(G)$-action on $Y \ltimes_\delta^{\mathcal{F}} G$ called the dual of δ.
   Similarly,

$$\widehat{\delta}\,(Y \overline{\ltimes}_\delta G) \subseteq (Y \overline{\ltimes}_\delta G)\,\overline{\otimes} L^\infty(G),$$

   i.e. $Y \overline{\ltimes}_\delta G$ is an $L^\infty(G)$-subcomodule of $(Y \ltimes_\delta^{\mathcal{F}} G, \widehat{\delta})$.

Saturation and non-degeneracy

Definition

Let (M, Δ) be a HvNa with $M \subseteq \mathcal{B}(K)$ and let (X, α) be an M-comodule with $X \subseteq \mathcal{B}(H)$. We say that (X, α) is non-degenerate if

$$X \overline{\otimes} \mathcal{B}(K) = \overline{\mathrm{span}}^{\mathrm{w}^*} \{(1_H \otimes b)\alpha(x) : \ b \in \mathcal{B}(K), \ x \in X\}.$$

On the other hand, (X, α) is called saturated if

$$\alpha(X) = \{y \in X \overline{\otimes}_{\mathcal{F}} M : \ (\mathrm{id}_X \otimes \Delta)(y) = (\alpha \otimes \mathrm{id}_M)(y)\}.$$

Saturation and non-degeneracy are necessary in order to establish a connection between the AP and Takesaki-duality as well as to characterize those comodules whose spatial and Fubini crossed products coincide.

$L^\infty(G)$-comodules are saturated and non-degenerate

Proposition (A., 2020)

For any locally compact group G every $L^\infty(G)$-comodule is both non-degenerate and saturated.

In particular, for an $L(G)$-comodule (Y, δ), we have that $(Y \ltimes_\delta^{\mathcal{F}} G, \widehat{\delta})$ is non-degenerate and this is the key to the proof of the following:

Corollary (A., 2020)

Let G be any locally compact group. For any $L(G)$-comodule (Y, δ), we have $Y \ltimes_\delta^{\mathcal{F}} G = Y \overline{\ltimes}_\delta G$.

Therefore we can simply write $Y \ltimes_\delta G$ instead of $Y \ltimes_\delta^{\mathcal{F}} G$ or $Y \overline{\ltimes}_\delta G$.

Takesaki-duality for $L^\infty(G)$-actions

Proposition (Hamana 2011, A. 2020)

For any $L^\infty(G)$*-comodule* (X,α) *we have*

$$\left(X \rtimes_\alpha^{\mathcal{F}} G\right) \ltimes_{\widehat{\alpha}} G = (X \overline{\rtimes}_\alpha G) \ltimes_{\widehat{\alpha}} G \simeq X \overline{\otimes} \mathcal{B}(L^2(G)).$$

Comment: Hamana proved that $X\overline{\otimes}\mathcal{B}(L^2(G)) \simeq \left(X \rtimes_\alpha^{\mathcal{F}} G\right) \ltimes_{\widehat{\alpha}}^{\mathcal{F}} G$ iff (X,α) is saturated (even for norm closed X!).
Similarly, we proved that $X\overline{\otimes}\mathcal{B}(L^2(G)) \simeq (X\overline{\rtimes}_\alpha G)\overline{\ltimes}_{\widehat{\alpha}} G$ iff (X,α) is non-degenerate.

Also, note that $\left(X \rtimes_\alpha^{\mathcal{F}} G\right) \ltimes_{\widehat{\alpha}} G = (X \overline{\rtimes}_\alpha G) \ltimes_{\widehat{\alpha}} G$ even though it may be $X \rtimes_\alpha^{\mathcal{F}} G \neq X \overline{\rtimes}_\alpha G$!

This means that an $L(G)$-comodule (Y,δ) cannot always be recovered from its crossed product $Y \ltimes_\delta G$.
Spoiler: unless G has the AP...

Regarding the AP as a stability property

If M is a von Neumann algebra, a net $\Phi_i \in CB_\sigma(M)$ is said to converge to the map $\Phi \in CB_\sigma(M)$ in the stable point-w*-topology if

$$(\mathrm{id}_{\mathcal{B}(\ell^2)} \otimes \Phi_i)(x) \xrightarrow{w^*} (\mathrm{id}_{\mathcal{B}(\ell^2)} \otimes \Phi)(x) \text{ for all } x \in \mathcal{B}(\ell^2)\overline{\otimes}M.$$

Equivalently, for any von Neumann algebra N, we have

$$(\mathrm{id}_N \otimes \Phi_i)(x) \xrightarrow{w^*} (\mathrm{id}_N \otimes \Phi)(x) \text{ for all } x \in N\overline{\otimes}M.$$

Every $u \in A(G) \simeq L(G)_*$ defines a map $M_u \in CB_\sigma(L(G))$ with $M_u(\lambda_s) = u(s)\lambda_s$.

$$M_u \colon L(G) \xrightarrow{\delta_G} L(G)\overline{\otimes}L(G) \xrightarrow{\mathrm{id}\otimes u} L(G)$$

Theorem (Haagerup-Kraus, 1993)

A locally compact group G has the AP if and only if there exists a net $\{u_i\}_{i\in I}$ in $A(G)$, such that $M_{u_i} \longrightarrow \mathrm{id}_{L(G)}$ in the stable point-w-topology.*

Saturated/non-degenerate $L(G)$-comodules & the AP

The stability property characterization of the AP can be reformulated as follows.

Proposition (A., 2020)

For a locally compact group G the following conditions are equivalent:

1. *G has the AP;*
2. *Every $L(G)$-comodule is saturated;*
3. *Every $L(G)$-comodule is non-degenerate;*
4. *All saturated $L(G)$-comodules are non-degenerate.*

The key to the proof is the observation that for any $L(G)$-comodule (Y, δ) we have

$$(\mathrm{id}_Y \otimes M_u) \circ \delta = \delta \circ (\mathrm{id}_Y \otimes u) \circ \delta \qquad \forall\, u \in A(G),$$

$$\begin{tikzcd}
Y \arrow[r, "\delta"] \arrow[dr, bend right, "\delta"'] & Y \overline{\otimes}_{\mathcal{F}} L(G) \arrow[r, "\mathrm{id}_Y \otimes u"] & Y \arrow[r, "\delta"] & Y \overline{\otimes}_{\mathcal{F}} L(G) \\
& Y \overline{\otimes}_{\mathcal{F}} L(G) \arrow[urr, bend right, "\mathrm{id}_Y \otimes M_u"'] & &
\end{tikzcd}$$

Comparing $X \rtimes_{\alpha}^{\mathcal{F}} G$ with $X \overline{\rtimes}_{\alpha} G$

Proposition (A., 2020)

For a locally compact group G and an $L^{\infty}(G)$-comodule (X, α) we have:

1. $\left(X \rtimes_{\alpha}^{\mathcal{F}} G, \widehat{\alpha}\right)$ *is a saturated* $L(G)$*-comodule;*
2. $(X \overline{\rtimes}_{\alpha} G, \widehat{\alpha})$ *is a non-degenerate* $L(G)$*-comodule.*

Theorem (A., 2020)

Let (X, α) be an $L^{\infty}(G)$-comodule. The following are equivalent:

1. $X \rtimes_{\alpha}^{\mathcal{F}} G = X \overline{\rtimes}_{\alpha} G$;
2. $(X \rtimes_{\alpha}^{\mathcal{F}} G, \widehat{\alpha})$ *is non-degenerate;*
3. $(X \overline{\rtimes}_{\alpha} G, \widehat{\alpha})$ *is saturated.*

The above yields an alternative proof of the Crann-Neufang theorem:
If G has the AP, then every $L(G)$-comodule is saturated (and non-degenerate) and thus we get $X \rtimes_{\alpha}^{\mathcal{F}} G = X \overline{\rtimes}_{\alpha} G$ for any $L^{\infty}(G)$-comodule (X, α).

Takesaki-type duality for $L(G)$-actions

Proposition (A., 2020)

For any $L(G)$-comodule (Y, δ), there is a w-continuous complete isometry*

$$\phi \colon Y \overline{\otimes} \mathcal{B}(L^2(G)) \to Y \overline{\otimes} \mathcal{B}(L^2(G)) \overline{\otimes} \mathcal{B}(L^2(G))$$

such that

1. (Y, δ) *is non-degenerate if and only if*
$$\phi\left(Y \overline{\otimes} \mathcal{B}(L^2(G))\right) = (Y \ltimes_\delta G) \,\overline{\rtimes}_{\widehat{\delta}} G;$$
2. (Y, δ) *is saturated if and only if*
$$\phi\left(Y \overline{\otimes} \mathcal{B}(L^2(G))\right) = (Y \ltimes_\delta G) \rtimes^{\mathcal{F}}_{\widehat{\delta}} G.$$

A dynamical characterization of the AP

Theorem (Crann-Neufang 2019, A. 2020)

For a locally compact group G the following conditions are equivalent:

(1) *G has the AP;*

(2) $(Y \ltimes_\delta G) \rtimes^{\mathcal{F}}_{\widehat{\delta}} G = (Y \ltimes_\delta G) \overline{\rtimes}_{\widehat{\delta}} G$ *for any $L(G)$-comodule (Y, δ);*

(3) $X \rtimes^{\mathcal{F}}_{\alpha} G = X \overline{\rtimes}_\alpha G$ *for any $L^\infty(G)$-comodule (X, α).*

Proof.

The implication (1) $\Longrightarrow$ (3) is already known by Crann-Neufang, while (3) $\Longrightarrow$ (2) is obvious, since $\left(Y \ltimes_\delta G, \widehat{\delta}\right)$ is an $L^\infty(G)$-comodule.

(2) $\Longrightarrow$ (1): It suffices to show that given (2) it follows that all saturated $L(G)$-comodules are non-degenerate.

Indeed, if (Y, δ) is saturated, then $\phi\left(Y \overline{\otimes} \mathcal{B}(L^2(G))\right) = (Y \ltimes_\delta G) \rtimes^{\mathcal{F}}_{\widehat{\delta}} G$ and thus from (2) it follows that $\phi\left(Y \overline{\otimes} \mathcal{B}(L^2(G))\right) = (Y \ltimes_\delta G) \overline{\rtimes}_{\widehat{\delta}} G$, which in turn means that (Y, δ) is non-degenerate. $\square$

A class of masa-bimodules arising as crossed products

The comultiplication $\delta_G \colon L(G) \to L(G)$ extends to an $L(G)$-action on $\mathcal{B}(L^2(G))$:

$$\delta_G(T) = W_G^*(T \otimes 1)W_G \qquad T \in \mathcal{B}(L^2(G)).$$

That is

$$\delta_G(f\lambda_s) = f\lambda_s \otimes \lambda_s$$

for any $f \in L^\infty(G)$ and $s \in G$.
For a closed ideal J of $A(G)$ with annihilator $J^\perp \subseteq L(G)$, Anoussis, Katavolos and Todorov introduced two 'different' $L^\infty(G)$-bimodules in $\mathcal{B}(L^2(G))$, namely

$$\mathrm{Bim}_{L^\infty(G)}(J^\perp) := \overline{\mathrm{span}}^{\mathrm{w}^*}\{L^\infty(G)J^\perp\},$$

$$(\mathrm{Sat}J)^\perp := \{T \in \mathcal{B}(L^2(G)) : \ S_u(T) = 0 \ \forall u \in J\},$$

where

$$S_u = (\mathrm{id} \otimes u) \circ \delta_G,$$

i.e. $S_u(f\lambda_s) = u(s)f\lambda_s$ for $f \in L^\infty(G), \ s \in G$.

$\mathrm{Bim}_{L^\infty(G)}(J^\perp) = (\mathrm{Sat}J)^\perp$

Theorem (Anoussis-Katavolos-Todorov, 2014)

$\mathrm{Bim}_{L^\infty(G)}(J^\perp) = (\mathrm{Sat}J)^\perp$ *for any closed ideal J of $A(G)$.*

A less technical proof of the above theorem can be obtained as follows:

- Since J is an ideal of $A(G)$, it follows that

$$\delta_G(J^\perp) \subseteq J^\perp \overline{\otimes}_{\mathcal{F}} L(G),$$

  i.e. $(J^\perp, \delta_G)$ is an $L(G)$-comodule.

- The canonical isomorphism

$$\Phi = \sigma \circ \delta_G : \mathcal{B}(L^2(G)) \to L(G) \ltimes_{\delta_G} G : \; f\lambda_s \mapsto \lambda_s \otimes f\lambda_s$$

  satisfies $\Phi(\mathrm{Bim}_{L^\infty(G)}(J^\perp)) = J^\perp \overline{\ltimes}_{\delta_G} G$ and $\Phi((\mathrm{Sat}J)^\perp) = J^\perp \ltimes_{\delta_G}^{\mathcal{F}} G$.

- Thus $\mathrm{Bim}_{L^\infty(G)}(J^\perp) = (\mathrm{Sat}J)^\perp$ because the Fubini and the spatial crossed products coincide for any $L(G)$-comodule.

The dual picture: $L(G)$-bimodules from ideals of $L^1(G)$

Similarly, for a closed (left) ideal J of $L^1(G)$ with annihilator $J^\perp \subseteq L^\infty(G)$ we have two $L(G)$-bimodules in $\mathcal{B}(L^2(G))$:

$$\mathrm{Bim}(J^\perp) := \overline{\mathrm{span}}^{\mathrm{w}^*}\{L(G)J^\perp\},$$

$$(\mathrm{Ran}J)^\perp := \{T \in \mathcal{B}(L^2(G)) : \ \Theta(h)(T) = 0 \ \forall h \in J\},$$

where

$$\Theta(h)(T) = \int_G h(s)\mathrm{Ad}\rho_s(T)\,\mathrm{d}s.$$

The fact that J is an ideal yields that

$$\alpha_G(J^\perp) \subseteq J^\perp \overline{\otimes} L^\infty(G)$$

thus $(J^\perp, \alpha_G)$ is an $L^\infty(G)$-comodule (i.e. a left translation invariant subspace of $L^\infty(G)$, since $\alpha_G \longleftrightarrow \mathrm{Ad}\lambda$) and there is a canonical isomorphism

$$\Psi\colon \mathcal{B}(L^2(G)) \to L^\infty(G) \rtimes_{\alpha_G} G : \ f\lambda_s \mapsto (1 \otimes \lambda_s)\alpha_G(f),$$

such that $J^\perp \rtimes^{\mathcal{F}}_{\alpha_G} G = \Psi((\mathrm{Ran}J)^\perp)$ and $J^\perp \overline{\rtimes}_{\alpha_G} G = \Psi(\mathrm{Bim}(J^\perp))$.

$\mathrm{Bim}(J^{\perp}) = (\mathrm{Ran}J)^{\perp}$?

Since $\mathrm{Bim}(J^{\perp}) \simeq J^{\perp}\overline{\rtimes}_{\alpha_G}G$ and $(\mathrm{Ran}J)^{\perp} \simeq J^{\perp} \rtimes^{\mathcal{F}}_{\alpha_G} G$, the Crann-Neufang theorem yields the following theorem first proved by Anoussis-Katavolos-Todorov for G being either abelian or compact or weakly amenable discrete.

Theorem (AKT 2018–CN 2019)

If G has the AP, then $\mathrm{Bim}(J^{\perp}) = (\mathrm{Ran}J)^{\perp}$ *for any closed left ideal J of $L^1(G)$.*

Is the AP necessary?

$$\begin{array}{ccc}
\mathcal{B}(L^2(G)) & \xrightarrow{\quad\Psi\quad} & L^{\infty}(G) \rtimes_{\alpha_G} G \\
\delta_G \big\downarrow & & \big\downarrow \widehat{\alpha_G} \\
\mathcal{B}(L^2(G))\overline{\otimes}L(G) & \xrightarrow{\Psi\otimes\mathrm{id}} & (L^{\infty}(G) \rtimes_{\alpha_G} G)\overline{\otimes}L(G)
\end{array}$$

Thus $\mathrm{Bim}(J^{\perp})$ and $(\mathrm{Ran}J)^{\perp}$ are $L(G)$-subcomodules of $(\mathcal{B}(L^2(G)), \delta_G)$ which are respectively isomorphic to $(J^{\perp}\overline{\rtimes}_{\alpha_G}G, \widehat{\alpha_G})$ and $(J^{\perp} \rtimes^{\mathcal{F}}_{\alpha_G} G, \widehat{\alpha_G})$.

Weaker (?) conditions that imply $\mathrm{Bim}(J^\perp) = (\mathrm{Ran}J)^\perp$

Corollary

Let J be a left closed ideal of $L^1(G)$. The following are equivalent:

1. $\mathrm{Bim}(J^\perp) = (\mathrm{Ran}J)^\perp$;
2. $(\mathrm{Bim}(J^\perp), \delta_G)$ *is a saturated $L(G)$-comodule;*
3. $((\mathrm{Ran}J)^\perp, \delta_G)$ *is a non-degenerate $L(G)$-comodule;*
4. $\mathrm{Bim}(J^\perp) = \{T \in \mathcal{B}(L^2(G)) : \ S_u(T) \in \mathrm{Bim}(J^\perp), \ \forall u \in A(G)\}$;
5. $(\mathrm{Ran}J)^\perp = \overline{\mathrm{span}}^{\mathrm{w}^*}\{S_u(T) : \ u \in A(G), \ T \in (\mathrm{Ran}J)^\perp\}$

Corollary

The following conditions are equivalent:

(i) *Every $L(G)$-subcomodule of $(\mathcal{B}(L^2(G)), \delta_G)$ is saturated;*

(ii) *Every $L(G)$-subcomodule of $(\mathcal{B}(L^2(G)), \delta_G)$ is non-degenerate;*

(iii) $T \in \overline{\{S_u(T) : \ u \in A(G)\}}^{w^*}$ *for any* $T \in \mathcal{B}(L^2(G))$.

Also, all above imply that $\mathrm{Bim}(J^\perp) = (\mathrm{Ran}J)^\perp$ *for any left closed ideal J of $L^1(G)$.*

Some food for thought...

Let $u \cdot T := S_u(T)$ for $u \in A(G)$ and $T \in \mathcal{B}(L^2(G))$.

$$
\begin{tikzcd}
G \text{ has the AP} \arrow[d, Rightarrow, shift right] & \\
T \in \overline{A(G)\cdot T}^{w^*}\ \forall T \in \mathcal{B}(L^2(G)) \arrow[u, Rightarrow, dashed, shift right, "?"'] \arrow[r, Rightarrow, shift left] \arrow[d, Rightarrow, shift right] & T \in \overline{A(G)\cdot T}^{w^*}\ \forall T \in L(G) \text{ (condition (H))} \arrow[l, Rightarrow, dashed, shift left, "?"] \arrow[d, Leftrightarrow] \\
\mathrm{Bim}(J^\perp) = (\mathrm{Ran}J)^\perp\ \forall J \lhd L^1(G) \arrow[u, Rightarrow, dashed, shift right, "?"'] & u \in \overline{A(G)u}^{\|\cdot\|}\ \forall u \in A(G) \text{ (property } D_\infty\text{)}
\end{tikzcd}
$$

Questions

1. Which of the implications above are strict?
2. Is property D_∞ necessary and/or sufficient in order to have $\mathrm{Bim}(J^\perp) = (\mathrm{Ran}J)^\perp$ for all (left) closed ideals $J \lhd L^1(G)$?
3. Are there any groups without property D_∞? What about $SL_3(\mathbb{Z})$?
4. $\exists\, G$ and J such that $\mathrm{Bim}(J^\perp) \neq (\mathrm{Ran}J)^\perp$?

Thanks for your attention!