The spectrum of the restriction to an invariant subspace

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08/05/2020

Preliminaries

Throughout X is a Banach space over $\mathbb{C}$ and $A : X \to X$ is a bounded linear operator.

Preliminaries

By $\sigma(A)$ we will denote the spectrum of A, i.e.

$$\sigma(A) = \{\lambda \in \mathbb{C} \,|\, A - \lambda I \text{ not invertible } \},$$

by $\rho(A)$ we will denote the resolvent of A, i.e.

$$\rho(A) = \mathbb{C} \setminus \sigma(A),$$

for each $\lambda \in \rho(A)$,

$$R(\lambda, A) = (A - \lambda I)^{-1},$$

and by $\sigma_{\mathrm{ap}}(A)$ we will denote the approximate point spectrum of A, i.e.

$$\sigma_{\mathrm{ap}}(A) = \{\lambda \in \mathbb{C} \,|\, \exists \{x_n\} \text{ such that } \|x_n\| = 1 \text{ and } (A - \lambda I)x_n \to 0\}.$$

Preliminaries

By M we will denote a closed subspace of X which is invariant under A, i.e. $A(M) \subseteq M$.
By $A|_M : M \to M$ we will denote the restriction of A on M.

Question

What can we say about the relation between $\sigma(A)$ and $\sigma\left(A|_M\right)$?

Two Examples

Example

Let $A \in \mathbb{M}_n(\mathbb{C})$ be an $n \times n$ matrix with eigenvalues $\{\lambda_1, \ldots, \lambda_n\}$ and corresponding eigenvectors $\{x_1, \ldots, x_n\}$.
If $M = \mathrm{span}\{x_k\}$, then obviously M is an invariant subspace of A, $A|_M = \lambda_k I_M$ and

$$\sigma\left(A|_M\right) = \{\lambda_k\} \subseteq \{\lambda_1, \ldots, \lambda_n\} = \sigma(A).$$

Two Examples

Example

Let $X = l^2(\mathbb{Z})$ and $A : X \to X$ be the bilateral shift, i.e.

$$A(\{\lambda_n\}) = \sum_{n=-\infty}^{\infty} \lambda_n e_{n+1} \,.$$

We know that

$$\sigma(A) = \{\lambda \in \mathbb{C} \,|\, |\lambda| = 1\} \,.$$

If

$$M = \{\{\lambda_n\} \in l^2(\mathbb{Z}) \,|\, \lambda_n = 0, \text{ for all } n < 0\} \,,$$

then M is an invariant subspace of A.
Moreover $A|_M$ is actually the unilateral shift on $l^2(\mathbb{N})$ and so

$$\sigma\left(A|_M\right) = \{\lambda \in \mathbb{C} \,|\, |\lambda| \leq 1\} \,.$$

Holes of the Spectrum

We will say that D is a hole of the spectrum $\sigma(A)$ if D is a bounded connected component of the resolvent set $\rho(A)$.
Recall that $\rho(A)$ has one unbounded connected component D_∞ and may have other connected components D.
Moreover recall that the resolvent function

$$\lambda \mapsto R(\lambda, A)$$

is analytic on each connected component of $\rho(A)$.

Filling the Holes of the Spectrum

Theorem

Let D be a connected component of $\rho(A)$.
If

$$D \cap \sigma\left(A|_M\right) \neq \emptyset\,,$$

then

$$D \subseteq \sigma\left(A|_M\right)\,.$$

Filling the Holes of the Spectrum

Corollary

Let D_∞ be the unbounded connected component of $\rho(A)$. Then

$$D_\infty \cap \sigma\left(A|_M\right) = \emptyset\,.$$

Filling the Holes of the Spectrum

Proof.

Assume that

$$D_\infty \cap \sigma\left(A|_M\right) \neq \emptyset .$$

Then, by the Theorem,

$$D_\infty \subseteq \sigma\left(A|_M\right) ,$$

which leads to a contradiction, since $\sigma\left(A|_M\right)$ is bounded while D_∞ is unbounded. So

$$D_\infty \cap \sigma\left(A|_M\right) = \emptyset .$$

□

Filling the Holes of the Spectrum

Corollary

Let D be a hole of $\sigma(A)$.
Then either

$$D \cap \sigma\left(A|_M\right) = \emptyset$$

or

$$D \subseteq \sigma\left(A|_M\right).$$

Proofs of the Theorem

Proof [J. Scroggs *[10, Lemmas 6–7, Theorem 4, Corollary 4.1]*; see also *[04, Theorem 1.29]*]

Proofs of the Theorem

Lemma

Let $T : X \to X$ be an invertible operator and M be an invertible subspace of T.
Then $T|_M$ is invertible if and only if M is an invariant subspace of T^{-1}.

Proof.

Exercise $\square$

Proofs of the Theorem

Let

$$\lambda_0 \in D \cap \sigma\left(A|_M\right) .$$

Then, by the previous Lemma,

$$(A - \lambda_0 I)^{-1}(M) \not\subset M .$$

Thus there exist $x \in M$ and $f \in M^{\perp}$ such that

$$\langle (A - \lambda_0 I)^{-1} x, f \rangle \neq 0 .$$

Recall that

$$\lambda \mapsto \langle (A - \lambda I)^{-1} x, f \rangle$$

is analytic on D.

Proofs of the Theorem

Therefore it is non-zero except on a discrete set.
If $D = D_\infty$, then that leads to a contradiction (Exercise: Why?)
If $D \neq D_\infty$, then D is contained in $\sigma\left(A|_M\right)$ except a discrete set.
Since $\sigma\left(A|_M\right)$ is closed that implies that $D \subseteq \sigma\left(A|_M\right)$.

Proofs of the Theorem

Proof [S. Parrott, 1960's; see *[03, Lemma 2]*, *[07, Problem 201]*, *[09, Theorem 0.8]*, *[02, Theorem II.2.11(c)]*]

Proofs of the Theorem

Lemma

Let $T : X \to X$ be a bounded linear operator and $\vartheta(\sigma(T))$ be the boundary of $\sigma(T)$.
Then $\vartheta(\sigma(T)) \subseteq \sigma_{\mathrm{ap}}(T)$.

Proof.

Exercise $\square$

Proofs of the Theorem

Lemma

We have that

$$\sigma_{\mathrm{ap}}\left(A|_M\right) \subseteq \sigma_{\mathrm{ap}}(A)\,.$$

Proof.

Let $\lambda \in \sigma_{\mathrm{ap}}\left(A|_M\right)$.
Then there exists a sequence $\{x_n\}$ in M such that $\|x_n\| = 1$ and

$$(A|_M - \lambda I_M)x_n \to 0$$

which obviously implies that

$$(A - \lambda I)x_n \to 0$$

and so $\lambda \in \sigma_{\mathrm{ap}}(A)$. $\square$

Proofs of the Theorem

Let $D^- = D \setminus \sigma\left(A|_M\right)$ and $D^+ = D \cap \sigma\left(A|_M\right)$.
Then $D^- \cup D^+ = D$ and $D^- \cap D^+ = \emptyset$.
Since D is open and $\sigma\left(A|_M\right)$ is closed, D^- is open.
On the other hand, if $\lambda \in D^+$, then $\lambda \in D$ and so $\lambda \notin \sigma(A)$.
In particular $\lambda \notin \sigma_{\mathrm{ap}}(A)$.
Thus, by the second Lemma, $\lambda \notin \sigma_{\mathrm{ap}}\left(A|_M\right)$.
So, by the first Lemma, $\lambda \notin \vartheta\left(\sigma\left(A|_M\right)\right)$.
Thus $\lambda \in \mathrm{int}\left(\sigma\left(A|_M\right)\right)$.
So $D^+ = D \cap \mathrm{int}\left(\sigma\left(A|_M\right)\right)$ and thus it is open.
Since both D^- and D^+ are open, $D^- \cup D^+ = D$ and $D^- \cap D^+ = \emptyset$, and D is connected we get that, since $D+ \neq \emptyset$, $D^+ = D$.

Proofs of the Theorem

Parrott's proof is more or less the same to the one given by T. Ito in *[08, Theorem 8]* for a normal operator A on a Hilbert space H. Ito's proof is a simplified version of the one given by J. Bram in *[01, Theorem 4]* again for a normal operator A on a Hilbert space H.

Proofs of the Theorem

Proof [N. Yannakakis and D. D. *[06]*]
Let $a \in D \cap \sigma(A|_M)$.
Assume that there exists $b \in D$ with $b \in \rho(A|_M)$.
Let C be any (continuous) rectifiable path that lies in D and connects a and b.

Proofs of the Theorem

First we show that there exists $c > 0$ such that

$$\|Ax - \lambda x\| \geq c\|x\|\,,\ \text{for all } \lambda \in C \text{ and } x \in M\,. \tag{1}$$

Assume the contrary, i.e. that there exists a sequence $\{\lambda_n\}$ in C and a sequence $\{x_n\}$ in M, with $\|x_n\| = 1$, such that $\|Ax_n - \lambda_n x_n\| \to 0$.
Then, since $\{\lambda_n\}$ is bounded, it has a subsequence, which for simplicity we denote again by $\{\lambda_n\}$, that converges to some $\lambda_0 \in C$ (note that C is the range of a continuous function and hence it is closed).
But then

$$\|Ax_n - \lambda_0 x_n\| \leq \|Ax_n - \lambda_n x_n\| + |\lambda_n - \lambda_0|$$

and hence $\|Ax_n - \lambda_0 x_n\| \to 0$, which is a contradiction, since $\lambda_0 \in \rho(A)$.

Proofs of the Theorem

As one may easily see inequality (1) implies that the resolvent function

$$\lambda \mapsto (A|_M - \lambda I_M)^{-1},$$

of the restriction $A|_M$, is bounded on $C \cap \rho(A|_M)$.
In particular

$$\|(A|_M - \lambda I_M)^{-1}\| \leq \frac{1}{c}, \text{ for all } \lambda \in C \cap \rho(A|_M).$$

Hence, by the elementary properties of the resolvent function, we get that if

$$|b - \lambda| < c \leq \frac{1}{\|R_b\|},$$

then $\lambda \in \rho(A|_M)$.

Proofs of the Theorem

Since c is independent of λ the above argument shows that if two λ's that belong to C are within c of each other and one is in the resolvent set of the restriction then the other is also in the resolvent set of the restriction. Therefore, if we divide the arc C into subarcs of length less than c, then the subarc containing b has the other endpoint in the resolvent set of the restriction, and so on.
The last subarc contains a and so a is also in the resolvent set of the restriction which is a contradiction.

A "Generalization"

Theorem

Let $A, B : X \to X$ be bounded linear operators such that

$$\sigma_{\mathrm{ap}}(B) \subseteq \sigma_{\mathrm{ap}}(A)$$

and D be a connected component of $\rho(A)$.
If

$$D \cap \sigma(B) \neq \emptyset\,,$$

then

$$D \subseteq \sigma(B)\,.$$

This "generalization" is mentioned by P. Halmos in *[07, Problem 201]*.
The proofs by Parrott and Yannakakis and D. can be used to prove it.

Other Similar Results

Similar results hold if A is a closed densely defined operator *[11]* as well as if instead of $A|M$ we take the operator $A_M : X/M \to X/M$ *[05]*.

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