Κατασκευές σταυρωτών γινομένων αλγεβρών τελεστών II

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Ημι-σταυρωτά γινόμενα (semicrossed products)

Έστω δυναμικό σύστημα (X,ϕ) όπου X συμπαγής Hausdorff και $\phi : X \to X$ συνεχής συνάρτηση. Θέτω $\mathscr{C} := C(X)$ και για κάθε $x \in X$ αναπαριστώ στον $H_x := \ell^2(\mathbb{Z}_+)$:

$$\pi_x(f) = \mathrm{diag}(f(\phi^n(x))) = \begin{bmatrix} f(x) & 0 & 0 & 0 & \dots \\ 0 & f(x_1) & 0 & 0 & \dots \\ 0 & 0 & f(x_2) & 0 & \dots \\ 0 & 0 & 0 & f(x_3) & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix},$$

(όπου $f \in \mathscr{C}$, $x_n = \phi^n(x)$). Επίσης γράφω:

$$S_x = \begin{bmatrix} 0 & 0 & 0 & 0 & \dots \\ 1 & 0 & 0 & 0 & \dots \\ 0 & 1 & 0 & 0 & \dots \\ 0 & 0 & 1 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Αθροίζω:

Ημι-σταυρωτά γινόμενα (semicrossed products)

$$\text{Ορίζω} \quad H := \oplus_{x\in X} H_x$$
$$\pi(f) := \oplus_{x\in X} \pi_x(f)$$
$$S := \oplus_{x\in X} S_x$$

Το ημι-σταυρωτό γινόμενο $\mathscr{C} \rtimes_\phi \mathbb{Z}_+$ είναι η κλειστή υπάλγεβρα (όχι *-υπάλγεβρα) της $\mathscr{B}(H)$ που παράγεται από τα $\{\pi(f) : f \in \mathscr{C}\} \cup \{S\}$.
Ελέγχεται ότι ικανοποιείται η 'covariance relation'

$$\pi(f)S = S\pi(f \circ \phi)$$

(οπότε $S\pi(f)S\pi(g) = S^2\pi(f\circ\phi)\pi(g) = S^2\pi((f\circ\phi)g)$ κ.λπ.).
Έπεται ότι το $\mathscr{C} \rtimes_\phi \mathbb{Z}_+$ είναι η κλειστή θήκη όλων των 'πολυωνύμων' $\sum_{n=0}^{N} S^n\pi(f_n)$ με συντελεστές $\pi(f_n)$ από την $\mathscr{C}$.

Παράδειγμα $X = \{x, y\}$

$$
\begin{aligned}
\pi(f) &= \pi_x(f) \oplus \pi_y(f) \\
&= \begin{bmatrix} f(x) & 0 & 0 & 0 \\ 0 & f(\phi(x)) & 0 & 0 \\ 0 & 0 & f(y) & 0 \\ 0 & 0 & 0 & f(\phi(y)) \end{bmatrix}, \\
S &= S_x \oplus S_y \\
&= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}.
\end{aligned}
$$

Αλλιώς:

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$$\pi'(f) = D(f) \oplus D(f \circ \phi) = \begin{bmatrix} f(x) & 0 \\ 0 & f(y) \end{bmatrix} \oplus \begin{bmatrix} f(\phi(x)) & 0 \\ 0 & f(\phi(y)) \end{bmatrix}$$

$$= \begin{bmatrix} f(x) & 0 & 0 & 0 \\ 0 & f(y) & 0 & 0 \\ 0 & 0 & f(\phi(x)) & 0 \\ 0 & 0 & 0 & f(\phi(y)) \end{bmatrix},$$

$$S = \begin{bmatrix} 0 & 0 \\ I & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}.$$

C*-dynamical systems

Definition

A C*-dynamical system (C*DS) is a pair consisting of a C*-algebra $\mathscr{C}$ (here unital, for simplicity) equipped with a *-endomorphism $\alpha : \mathscr{C} \to \mathscr{C}$. When α is bijective (i.e. an automorphism) we call $(\mathscr{C}, \alpha)$ *reversible*.

Examples

• $\mathscr{C} = \mathbb{C}$, action α: trivial.
• $\mathscr{C} = \ell^\infty \hookrightarrow \mathscr{B}(\ell^2)$ as diagonal operators, action α: shift (?).
• $\mathscr{C} = C(\mathbb{T}) \hookrightarrow \mathscr{B}(L^2(\mathbb{T}))$ as multiplication operators, action α: (irrational) rotation. (• *But can consider non-commutative* $\mathscr{C}$.)

Definition

A covariant pair for the C*DS $(\mathscr{C}, \alpha)$ is a pair (π, U) consisting of a *-representation $\pi : \mathscr{C} \to \mathscr{B}(H)$ and a unitary U on the same space H satisfying the covariance condition

$$\pi(x)U = U\pi(\alpha(x)) \quad \text{for all } x \in \mathscr{C}. \qquad \text{(Icov)}$$

Existence of covariant representations

May assume $\mathscr{C} \subseteq \mathscr{B}(H_0)$. *There always exists a covariant representation* (π, U) *which is faithful on* $\mathscr{A}_0$*.*

Idea: 'Enlarge' the space (if necessary) to accomodate a U so that $\pi(\alpha(c)) = U^*\pi(c)U \ \forall c \in \mathscr{C}$ holds.
Consider

$$H = \ell^2(\mathbb{Z}) \otimes H_0 := \{(\xi(n))_{n\in\mathbb{Z}} : \xi(n) \in H_0 \ \forall n, \sum_n \|\xi(n)\|^2_{H_0} < \infty\}$$

$$\langle(\xi(n)), (\eta(n))\rangle := \sum_n \langle \xi(n), \eta(n)\rangle_{H_0}$$

and $U_0 : H \to H$:

$$U_0 : (\ldots, \xi(-2), \xi(-1), \boxed{\xi(0)}, \xi(1), \xi(2), \ldots)$$
$$\to (\ldots, \xi(-3), \xi(-2), \boxed{\xi(-1)}, \xi(0), \xi(1), \ldots)$$

Existence of covariant representations

Represent as matrices with entries in $\mathscr{B}(H_0)$,

$$\pi_0(c) = \mathrm{diag}(\alpha^n(c)) = \begin{bmatrix} \ddots & & & & \\ & \alpha^{-1}(c) & & & \\ & & \boxed{c} & & \\ & & & \alpha(c) & \\ & & & & \ddots \end{bmatrix},$$

$$U_0 = \begin{bmatrix} \ddots & & & & & \\ \ddots & 0 & & & & \\ & \mathbf{1}_{H_0} & \boxed{0} & & & \\ & & \mathbf{1}_{H_0} & 0 & & \\ & & & \mathbf{1}_{H_0} & \ddots & \\ & & & & \ddots & \end{bmatrix}.$$

The reduced C*-crossed product

We proved that the seminorm

$$\left\|\sum_k u^k c_k\right\|_r := \left\|(U_0 \times \pi_0)\left(\sum_k u^k c_k\right)\right\|_{\mathscr{B}(H)} = \left\|\sum_k U_0^k \pi_0(c_k)\right\|_{\mathscr{B}(H)}$$

is in fact a norm on $\mathscr{A}_0$, and is clearly an algebra seminorm satisfying the C*-condition. Therefore the completion of $\mathscr{A}_0$ in this norm is a C*-algebra.

Definition

The C*-reduced crossed product $\mathscr{C} \rtimes_{\alpha,r} \mathbb{Z}$ associated to the C*DS $(\mathscr{C}, \alpha)$ is the completion of $\mathscr{A}_0$ in the norm $\|\cdot\|_r$. Equivalently, it is the concrete C*-subalgebra of $\mathscr{B}(H_0 \otimes \ell^2(\mathbb{Z}))$ generated by $\pi_0(\mathscr{C})$ and U_0; it is the closure of $\mathscr{A}_0$ in the norm of $\mathscr{B}(H)$.

Examples

• $\mathscr{C} = \mathbb{C}$, action α: trivial:
$\mathscr{A}_0$: trigonometric polynomials, $\mathscr{C} \rtimes_{\alpha,r} \mathbb{Z} = C(\mathbb{T})$.

• $\mathscr{C} = \ell^\infty \hookrightarrow \mathscr{B}(\ell^2)$ as diagonal operators, action α: shift.
$\mathscr{C} \rtimes_{\alpha,r} \mathbb{Z} \hookrightarrow \mathscr{B}(\ell^2)$ is called the uniform Roe algebra.

• $\mathscr{C} = C(\mathbb{T}) \hookrightarrow \mathscr{B}(L^2(\mathbb{T}))$ as multiplication operators, action α: irrational rotation. $\mathscr{C} \rtimes_{\alpha,r} \mathbb{Z} = C^*(U, V)$ ("*universal*") where U, V unitaries satisfying $VU = \lambda UV$ (Weyl).

The reduced C*-crossed product

Definition

The C*-reduced crossed product $\mathscr{C} \rtimes_{\alpha,r} \mathbb{Z}$ associated to the C*DS $(\mathscr{C}, \alpha)$ is the completion of $\mathscr{A}_0$ in the norm $\|\cdot\|_r$. Equivalently, it is the concrete C*-subalgebra of $\mathscr{B}(H)$ generated by $\pi_0(\mathscr{C})$ and U_0; it is the closure of $\mathscr{A}_0$ in the norm of $\mathscr{B}(H)$.

Fact The reduced crossed product $\mathscr{C} \rtimes_{\alpha,r} \mathbb{Z}$ does not depend on the way $\mathscr{C}$ is represented: if the identity representation $\mathscr{C} \to \mathscr{B}(H_0)$ is replaced by *any faithful* representation $\pi_1 : \mathscr{C} \to \mathscr{B}(H_1)$ then the resulting norm $\|\cdot\|_{1,r}$ coincides with $\|\cdot\|_r$ on $\mathscr{A}_0$.

The full C*-crossed product

... but recall that we wanted the full crossed product to 'encode' *all* covariant pairs (U, π), so we defined, for $p \in \mathscr{A}_0$,

$$\|p\|_* = \sup\{\|(U \times \pi)(p)\| : \text{all covariant pairs } (\pi, U)\}.$$

(Recall $\quad (U \times \pi)\left(\sum_k u^k c_k\right) = \sum_k U^k \pi(c_k)$

Since clearly $\left\|\sum_k u^k c_k\right\|_r \leq \left\|\sum_k u^k c_k\right\|_*$, we have also shown that the seminorm $\|\cdot\|_*$ *is also a norm on* $\mathscr{A}_0$.

Definition

The full crossed product $\mathscr{C} \rtimes_\alpha \mathbb{Z}$ of the C*DS $(\mathscr{C}, \alpha)$ is the completion of the covariance algebra $\mathscr{A}_0$ in the norm $\|\cdot\|_*$.

Corollary (Universal property)

There is a bijective correspondence between covariant pairs (π, U) *and representations of the C*-algebra* $\mathscr{C} \rtimes_\alpha \mathbb{Z}$.

The full C*-crossed product

Since $\left\|\sum_k u^k c_k\right\|_r \leq \left\|\sum_k u^k c_k\right\|_*$, the identity map

$$(\mathscr{A}_0, \|\cdot\|_*) \longrightarrow (\mathscr{A}_0, \|\cdot\|_r)$$

is contractive, hence extends to a contraction (and also a *-morphism)

$$\lambda : \mathscr{C} \rtimes_\alpha \mathbb{Z} \longrightarrow \mathscr{C} \rtimes_{\alpha,r} \mathbb{Z}$$

which is onto (why?).
But is this map 1-1 *on the full crossed product*?
In general, not necessarily (for example, when we have an action of $\mathbb{F}_2$ instead of $\mathbb{Z}$).
But in the case of $\mathbb{Z}$, the answer is YES!

Injectivity of λ

If $a \in \mathscr{C} \rtimes_\alpha \mathbb{Z}$ and $\lambda(a) = 0$, need to show $a = 0$.
In case $a = \sum_k u^k c_k$ is in $\mathscr{A}_0$, we have

$$0 = \lambda(a) = \sum_k U_0^k \pi_0(c_k) \overset{inj}{\Rightarrow} c_k = 0 \, \forall k \Rightarrow a = 0$$

(inj): Το έχουμε δείξει.
But how to find the 'Fourier coefficients' c_k for general $a \in \mathscr{A} := \mathscr{C} \rtimes_\alpha \mathbb{Z}$?
And secondly, if all Fourier coefficients of some $a \in \mathscr{A}$ are 0, does it follow that $a = 0$?

Fourier coefficients

For $k \in \mathbb{Z}$, define

$$E_k : \mathscr{A}_0 \to \mathscr{C} : \sum_n u^n c_n \to c_k.$$

Clearly linear. Also $\|\cdot\|_*$-contractive: for $\xi, \eta \in H_0$ of norm one,

$$\langle c_m \xi, \eta \rangle_{H_0} = \left\langle \sum_k U_0^k \pi_0(c_k)(e_0 \otimes \xi), (e_m \otimes \eta) \right\rangle_H$$

$$\Rightarrow \left| \langle c_m \xi, \eta \rangle_{H_0} \right| \leq \left\| \sum_k U_0^k \pi_0(c_k) \right\|_{\mathscr{B}(H)} = \left\| \sum_n u^n c_n \right\|_r \leq \left\| \sum_n u^n c_n \right\|_*$$

$$\Rightarrow \|E_m(a)\|_{\mathscr{C}} = \|c_m\|_{\mathscr{C}} \leq \|a\|_r \leq \|a\|_* \quad \forall a \in \mathscr{A}_0$$

Fourier coefficients

Thus E_k extends to a linear contraction on the $\|\cdot\|_*$-completion:

$$E_k : \mathscr{C} \rtimes_\alpha \mathbb{Z} \to \mathscr{C}$$

But if $\lambda(a) = 0$, i.e. $\|\lambda(a)\|_r = 0$ then $\|E_k(a)\|_{\mathscr{C}} \leq \|\lambda(a)\|_r = 0$ for all $k \in \mathbb{Z}$. So all we have to prove is the

Claim If $E_k(a) = 0$ for all $k \in \mathbb{Z}$ then $a = 0$.

: injectivity of the ‘Fourier transform’!

Locating $E_k(a)$

Define *dual or gauge* action (of $\mathbb{T}$) first on $\mathscr{A}_0$: for $e^{it} \in \mathbb{T}$, let

$$\theta_t\left(\sum_n u^n c_n\right) = \sum_n (e^{it}u)^n c_n$$

Claim. Each θ_t extends to an isometric *-automorphism of $\mathscr{C} \rtimes_\alpha \mathbb{Z}$.

Proof For each *-rep $\rho = U \times \pi$ of $\mathscr{A}_0$, $\rho \circ \theta_t$ is another. Hence $\|\rho(\theta_t(a))\| \leq \|a\|_*$ when $a \in \mathscr{A}_0$. Taking sup, $\|\theta_t(a)\|_* \leq \|a\|_*$.

Thus θ defines an action of the group $\mathbb{T}$ on $\mathscr{C} \rtimes_\alpha \mathbb{Z}$.
The group $\{\theta_t : e^{it} \in \mathbb{T}\}$ is called the dual automorphism group.

Now we calculate, first when $a \in \mathscr{A}_0$ and then for general $a \in \mathscr{A}$,

$$\frac{1}{2\pi}\int_0^{2\pi} \theta_t(a)e^{-imt}dt = u^m E_m(a).$$

Finally...

Proposition

Each $a \in \mathscr{C} \rtimes_\alpha \mathbb{Z}$ belongs to the $\|\cdot\|_*$-closed linear span of

$$\{u^k E_k(a) : k \in \mathbb{Z}\}.$$

Proof ... Hahn-Banach and injectivity of the usual Fourier transform on $C(\mathbb{T})$!

So, if all $E_k(a)$ vanish, then a must vanish.
We have shown that if $a \in \mathscr{A}$ and $\|a\|_r = 0$, then $a = 0$.
Therefore the map

$$\lambda : \mathscr{C} \rtimes_\alpha \mathbb{Z} \longrightarrow \mathscr{C} \rtimes_{\alpha,r} \mathbb{Z}$$

is injective, hence an isometric isomorphism! □

¡Muchas gracias, hasta la proxima!