Κατασκευές σταυρωτών γινομένων (crossed products) αλγεβρών τελεστών

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Παράδειγμα

Αναπαριστώ [1] τον ℓ^∞ ως διαγώνιους τελεστές $H := \ell^2(\mathbb{Z}_+)$:

$$D(c) := \begin{bmatrix} c_0 & 0 & 0 & 0 & \dots \\ 0 & c_1 & 0 & 0 & \dots \\ 0 & 0 & c_2 & 0 & \dots \\ 0 & 0 & 0 & c_3 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}, \quad c \in \ell^\infty,$$

και θεωρώ τον

$$S = \begin{bmatrix} 0 & 0 & 0 & 0 & \dots \\ 1 & 0 & 0 & 0 & \dots \\ 0 & 1 & 0 & 0 & \dots \\ 0 & 0 & 1 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

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Παράδειγμα

$$D(c) + SD(a) + S^2 D(b) := \begin{bmatrix} c_0 & 0 & 0 & 0 & \dots \\ a_0 & c_1 & 0 & 0 & \dots \\ b_0 & a_1 & c_2 & 0 & \dots \\ 0 & b_1 & a_2 & c_3 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Ποιά είναι η [κλειστή] γραμμική θήκη της άλγεβρας των «πολυωνύμων» $\sum_{k=0}^{N} S^k D(c_k)$;

Ημι-σταυρωτά γινόμενα (semicrossed products)

Έστω δυναμικό σύστημα (X,ϕ) όπου X συμπαγής Hausdorff και $\phi : X \to X$ συνεχής συνάρτηση. Θέτω $\mathscr{C} := C(X)$ και για κάθε $x \in X$ αναπαριστώ στον $H_x := \ell^2(\mathbb{Z}_+)$:

$$\pi_x(f) = \mathrm{diag}(f(\phi^n(x)) = \begin{bmatrix} f(x) & 0 & 0 & 0 & \dots \\ 0 & f(x_1) & 0 & 0 & \dots \\ 0 & 0 & f(x_2) & 0 & \dots \\ 0 & 0 & 0 & f(x_3) & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix},$$

(όπου $f \in \mathscr{C}$, $x_n = \phi^n(x)$). Επίσης γράφω:

$$S_x = \begin{bmatrix} 0 & 0 & 0 & 0 & \dots \\ 1 & 0 & 0 & 0 & \dots \\ 0 & 1 & 0 & 0 & \dots \\ 0 & 0 & 1 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Αθροίζω:

Ημι-σταυρωτά γινόμενα (semicrossed products)

$$\text{Ορίζω}\quad H := \oplus_{x\in X} H_x$$
$$\pi(f) := \oplus_{x\in X}\pi_x(f)$$
$$S := \oplus_{x\in X} S_x$$

Το ημι-σταυρωτό γινόμενο $\mathscr{C} \rtimes_\phi \mathbb{Z}_+$ είναι η κλειστή υπάλγεβρα (όχι *-υπάλγεβρα) της $\mathscr{B}(H)$ που παράγεται από τα $\{\pi(f) : f \in \mathscr{C}\} \cup \{S\}$.
Ελέγχεται ότι ικανοποιείται η 'covariance relation'

$$\pi(f)S = S\pi(f\circ\phi)$$

(οπότε $S\pi(f)S\pi(g) = S^2\pi(f\circ\phi)\pi(g) = S^2\pi((f\circ\phi)g)$ κ.λπ.).
Έπεται ότι το $\mathscr{C} \rtimes_\phi \mathbb{Z}_+$ είναι η κλειστή θήκη όλων των 'πολυωνύμων' $\sum_{n=0}^{N} S^n\pi(f_n)$ με συντελεστές $\pi(f_n)$ από την $\mathscr{C}$.

Σταυρωτά γινόμενα (crossed products)

Αν ϕ ομοιομορφισμός μπορώ να βάλω κάθε $H_x := \ell^2(\mathbb{Z})$, και $\pi_x(f) = \mathrm{diag}(f(\phi^n(x)), n \in \mathbb{Z}$ και στη θέση του S_x το bilateral shift

$$\pi_x(f) = \begin{bmatrix} \ddots & & & & \\ & f(x_{-1}) & & & \\ & & \boxed{f(x)} & & \\ & & & f(x_1) & \\ & & & & \ddots \end{bmatrix} \quad S_x = \begin{bmatrix} \ddots & & & & \\ \ddots & 0 & & & \\ & 1 & \boxed{0} & & \\ & & 1 & 0 & \\ & & & 1 & \ddots \\ & & & & \ddots \end{bmatrix}.$$

Το σταυρωτό γινόμενο $\mathscr{C} \rtimes_\phi \mathbb{Z}$ είναι η κλειστή *-υπάλγεβρα της $\mathscr{B}(H)$ που παράγεται από τα $\{\pi(f) : f \in \mathscr{C}\} \cup \{S\}$. Είναι η κλειστή θήκη όλων των 'τριγωνομετρικών πολυωνύμων' $\sum_{n=-N}^{N} S^n \pi(f_n)$ με συντελεστές $\pi(f_n)$ από την $\mathscr{C}$.

Απ′ την αρχή: Dynamical Systems

A commutative dynamical system (CDS) is a pair (X,ϕ) where X is a set (ex: $X \subseteq \mathbb{R}^n$) and $\phi : X \to X$ is a self-map.
So we have an action

$$\mathbb{Z}_+ \curvearrowright X : \; 0 \to id, n \to \phi^n := \phi \circ \phi \cdots \circ \phi$$

or $\mathbb{Z} \curvearrowright X$ ($-n \to \phi^{-1} \circ \phi^{-1} \cdots \circ \phi^{-1}$) when ϕ is bijective (i.e. the system is *reversible*).

When X is a compact or locally compact space and ϕ is continuous then (X,ϕ) is called a topological dynamical system (TDS) (reversible TDS when ϕ is a homeomorphism).

When X is a measure space (or a probability space) and ϕ (and its inverse, if it exists) is *measurable and measure-preserving* [2] then (X,ϕ) is called a measurable dynamical system (MDS).

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[2] more generally, measure-class preserving

Dynamical Systems

When we have more than one map on X, say $\{\phi_a, \phi_b\}$, we speak of a *multivariable dynamical system*.
Here we have an action

$$\mathbb{F}_+^2 \curvearrowright X$$
$$aba^2b \to \phi_b \circ \phi_a \circ \phi_a \circ \phi_b \circ \phi_a$$

[Davidson-Katsoulis, Kakariadis-Katsoulis, ...]
(or an action $\mathbb{Z}_+^2 \curvearrowright X$ if the maps commute ($\phi_a \circ \phi_b = \phi_b \circ \phi_a$).

More generally we could study a DS (X, G) where G is a *group* of (bijective) maps $g : X \to X$ (i.e. the group law is given by composition: $g_1 \circ g_2$).

Classical or commutative systems

From now on, let X be compact T_2 (for simplicity) (or even a metric space) and $\phi : X \to X$ continuous.

The action $\phi : X \to X$ can be transferred to an action

$$\alpha : C(X) \to C(X) : f \to f \circ \phi.$$

Advantage: $C(X)$ is a *linear algebra* and α preserves its structure. [3]

Exercise

α is 1-1 iff ϕ is onto; α is onto iff ϕ is 1-1.

The action is transferred from the state space X to the observables (functions on X):

$$(X, \phi) \rightsquigarrow (C(X), \alpha).$$

[3] In the MDS case, transfer the action $\phi : X \to X$ to an action $\beta : L^\infty(X) \to L^\infty(X) : f \to f \circ \phi$.

Quantum or non-commutative systems

$$(X, \phi) \rightsquigarrow (C(X), \alpha).$$

Thus the action α is transferred from the *state space* to the *observables*.
In Quantum Mechanics *states* correspond to unit vectors [or rays...] in a Hilbert space H; the *observables* define self-adjoint operators on H and the *dynamics* (for a reversible system) define an action $U_t : H \to H\,(t \in \mathbb{R})$ -time development- on the state space, where the U_t preserve the linear structure and the length: they are unitary operators.

Again we may transfer the action from the state space to the (possibly non-commutative) C*-algebra $\mathscr{C} \subseteq \mathscr{B}(H)$ generated by the observables:

$$U_t : H \to H \qquad \rightsquigarrow \qquad \alpha_t : \mathscr{C} \to \mathscr{C} : T \to U_t^{-1} T U_t$$

Schrödinger picture Heisenberg picture

C*-dynamical systems

Definition

A C*-dynamical system (C*DS) is a pair consisting of a C*-algebra $\mathscr{C}$ (here unital, for simplicity) equipped with a *-endomorphism $\alpha : \mathscr{C} \to \mathscr{C}$. When α is bijective (i.e. an automorphism) we call $(\mathscr{C}, \alpha)$ *reversible*.

Example

Let $\mathscr{C} \subseteq \mathscr{B}(H)$ be a C*-subalgebra and $U \in \mathscr{B}(H)$ an isometry such that $U^*\mathscr{C}U \subseteq \mathscr{C}$. Then $\alpha(x) = U^*xU$ is a *-endomorphism; if U is unitary ($U^{-1} = U^*$) then $\alpha \in Aut(\mathscr{C})$.
We say that α is *spatial, implemented* by U.

Question

More generally, given a (C*DS) $(\mathscr{C}, \alpha)$, can we find a *-representation $\pi : \mathscr{C} \to \mathscr{B}(H)$ and $U \in \mathscr{B}(H)$ so that:

$$\pi(\alpha(c)) = U^*\pi(c)U \quad \forall c \in \mathscr{C}? \tag{cov}$$

An example

Let $\mathscr{C} = C(X)$ and let ϕ be a homeomorphism. Suppose X supports a prob. measure μ which is ϕ-invariant, i.e. $\mu(\phi^{-1}(E)) = \mu(E)$ for every Borel $E \subseteq X$. Let $H = L^2(X,\mu)$ and represent $\mathscr{C}$ by defining $\pi(f), f \in C(X)$ as follows: [4]

$$\pi(f)\xi = f\xi \quad \xi \in H.$$

The operator U defined on H by

$$U(\xi) = \xi \circ \phi^{-1}$$

is unitary (Exercise!), and

$$\pi(\alpha(f))U^*\xi = U^*\pi(f)\xi \quad \text{for all } \xi \in H.$$

Proof:

$$\pi(\alpha(f))U^* : \xi \xrightarrow{U^*} \xi \circ \phi \xrightarrow{\pi(\alpha(f))} \alpha(f)(\xi \circ \phi) = (f \circ \phi)(\xi \circ \phi)$$

$$U^*\pi(f) : \xi \xrightarrow{\pi(f)} f\xi \xrightarrow{U^*} (f\xi) \circ \phi = (f \circ \phi)(\xi \circ \phi)$$

[4] Suppose $\mu(U) > 0$ for every open $U \subseteq X$ to make π injective.

Ένα συγκεκριμένο παράδειγμα

Έστω $X = \mathbb{T} = \{e^{it} : t \in [0, 2\pi]\}$ και $\phi(e^{it}) = e^{i(t+\theta)}$ όπου $\theta/2\pi$ άρρητος. Θέτουμε $H = L^2(\mathbb{T}, \mu)$ (μέτρο Lebesgue).
Η αναπαράσταση π παράγεται απ' την εικόνα του $\pi(\zeta)$ (όπου $\zeta(e^{it}) = e^{it}$). Αν $\zeta \in C(X)$ είναι η συνάρτηση $\zeta(e^{it}) = e^{it}$, γράφω $\pi(\zeta) = V$, δηλαδή:

$$(V\xi)(z) = z\xi(z) \quad \xi \in H_2, z = e^{it} \in \mathbb{T}.$$

Ο V είναι unitary. Επίσης ο unitary τελεστής U ορίζεται από

$$(U\xi)(z) = \xi(\bar{\lambda} z) \quad (\text{όπου } \lambda = e^{i\theta})$$

Η covariance condition γράφεται ισοδύναμα

$$VU = \lambda UV$$

($\sim$ η σχέση Weyl της Κβαντομηχανικής).

The full or universal C*-crossed product

Idea: Given $(\mathscr{C}, \alpha)$, to form a 'larger' C*-algebra $\mathscr{A} = \mathscr{C} \rtimes_\alpha \mathbb{Z}$ containing $\mathscr{C}$ as well as a unitary element u in such a way that the covariance condition $\alpha(c) = u^* c u$ holds in $\mathscr{A}$.

We will define a *-algebra $\mathscr{A}_0$, define a suitable C*-norm on it and complete to get $\mathscr{A}$.

(a) The covariance algebra. First form the linear space $\mathscr{A}_0$ of all 'Laurent polynomials' p in one variable u with coefficients in $\mathscr{C}$:

$$p(u) = \sum_{k=-n}^{n} u^k c_k, \qquad c_k \in \mathscr{C}.$$

Make $\mathscr{A}_0$ into a *-algebra:

$$(pq)(u) = \left(\sum_k u^k c_k\right)\left(\sum_m u^m d_m\right) = \sum_{k,m} u^k c_k u^m d_m = ?$$

(a) The covariance algebra $\mathscr{A}_0$

Want $\alpha(c) = u^* c u$ or $cu = u\alpha(c)$, hence $cu^m = u^m \alpha^m(c)$. So define multiplication by

$$\left(\sum_k u^k c_k\right)\left(\sum_m u^m d_m\right) = \sum_{k,m} u^{k+m} \alpha^m(c_k) d_m = \sum_n u^n \sum_m \alpha^m(c_{n-m}) d_m$$

Similarly, want $cu^{-k} = u^{-k}\alpha^{-k}(c)$ so define

$$\left(\sum_k u^k c_k\right)^* = \sum_k (u^k c_k)^* = \sum_k c_k^* u^{-k} = \sum_k u^{-k} \alpha^{-k}(c_k^*) = \sum_n u^n \alpha^n(c_{-n}^*)$$

(b) Covariant representations

Want to represent $\mathscr{A}_0$ by bounded operators on Hilbert space. Observe that any *-representation

$$\rho : \mathscr{A}_0 \to \mathscr{B}(H)$$

defines, by restriction, a representation $\rho_c : \mathscr{C} \to \mathscr{B}(H)$ and a unitary $V \in \mathscr{B}(H)$ such that

$$\rho\left(\sum_k u^k x_k\right) = \sum_k V^k \rho_c(x_k)\,.$$

Note that the covariance condition

$$\rho_c(x)V = V\rho_c(\alpha(x)) \quad \text{for all } \ x \in \mathscr{C}$$

holds.

(b) Covariant representations and repr. of $\mathscr{A}_0$

Conversely, suppose given $\pi : \mathscr{C} \to \mathscr{B}(H)$ and $U \in \mathscr{B}(H)$ (same H!) satisfying the covariance condition

$$\pi(x)U = U\pi(\alpha(x)) \quad \text{for all } x \in \mathscr{C}. \tag{lcov}$$

This is the *left covariance condition*. Then we define

$$U \times \pi : \mathscr{A}_0 \to \mathscr{B}(H)$$
$$(U \times \pi)\left(\sum_k u^k c_k\right) = \sum_k U^k \pi(c_k)$$

It can be readily verified that this (clearly linear) map is in fact a *-representation of $\mathscr{A}_0$ on H.

Covariant pairs

Definition

A covariant pair for the C*DS $(\mathscr{C}, \alpha)$ is a pair (π, U) consisting of a *-representation $\pi : \mathscr{C} \to \mathscr{B}(H)$ and a unitary U on the same space H satisfying the covariance condition

$$\pi(x)U = U\pi(\alpha(x)) \quad \text{for all } \ x \in \mathscr{C}. \tag{Icov}$$

We have shown the

Proposition

*There is a bijective correspondence between covariant representations (π, U) of $(\mathscr{C}, \alpha)$ and *-representations $U \times \pi$ of the covariance algebra $\mathscr{A}_0$.*

(c) Completing the covariance algebra

To obtain a C*-algebra, we need to complete $\mathscr{A}_0$ with respect to an algebra norm satisfying the C*-condition. To 'encode' all covariant pairs, define, for $p = \sum_k u^k c_k \in \mathscr{A}_0$

$$\begin{aligned} \|p\|_* &= \sup\{\|\rho(p)\| : \text{all *-reps. } (\rho, H) \text{ of } \mathscr{A}_0\} \\ &= \sup\{\|(U \times \pi)(p)\| : \text{all covariant pairs } (\pi, U)\}. \end{aligned}$$

This is finite, since every representation is $\|\cdot\|_1$-contractive:

$$\left\| (U \times \pi)\left(\sum_k u^k c_k\right) \right\| \le \sum_k \left\| U^k \pi(c_k) \right\| \le \sum_k \|c_k\|$$

It is also easy to verify that it is an algebra *seminorm* and that it satisfies the C*-condition.

But are there any covariant representations?

Existence of covariant representations

We know (Gelfand-Naimark) that every C*-algebra $\mathscr{C}$ admits a faithful (i.e. 1-1) representation on some Hilbert space, so may assume $\mathscr{C} \subseteq \mathscr{B}(H_0)$. But is there always a covariant representation (π, U) which is faithful on $\mathscr{A}_0$?

Idea: 'Enlarge' the space (if necessary) to accomodate a U so that $\pi(\alpha(c)) = U^*\pi(c)U \ \forall c \in \mathscr{C}$ holds.
Consider

$$H = \ell^2(\mathbb{Z}) \otimes H_0 := \{(\xi(n))_{n\in\mathbb{Z}} : \xi(n) \in H_0 \ \forall n, \sum_n \|\xi(n)\|^2_{H_0} < \infty\}$$

$$\langle(\xi(n)), (\eta(n))\rangle := \sum_n \langle \xi(n), \eta(n)\rangle_{H_0}$$

and $U_0 : H \to H$:

$$\begin{aligned} U_0 : &(\dots, \xi(-2), \xi(-1), \boxed{\xi(0)}, \xi(1), \xi(2), \dots) \\ &\to (\dots, \xi(-3), \xi(-2), \boxed{\xi(-1)}, \xi(0), \xi(1), \dots) \end{aligned}$$

Existence of covariant representations

Notation: for $n \in \mathbb{Z}$ and $\xi \in H_0$ denote by $e_n \otimes \xi \in H$ the function

$$\mathbb{Z} \to H_0 : m \to (e_n \otimes \xi)(m) = \begin{cases} \xi, & m = n \\ 0, & m \neq n \end{cases}$$

(note $H = \overline{\mathrm{span}}\{e_n \otimes \xi : n \in \mathbb{Z}, \xi \in H_0\}$). The map U_0 is given by

$$U_0(e_n \otimes \xi) = e_{n+1} \otimes \xi .$$

Also define the representation $\pi_0 : \mathscr{C} \to \mathscr{B}(H)$ by

$$\pi_0(c)(e_n \otimes \xi) = e_n \otimes \alpha^n(c)\xi$$

where $c \in \mathscr{C}, \xi \in H_0, n \in \mathbb{Z}$.

Existence of covariant representations

Representing these as matrices with entries in $\mathscr{B}(H_0)$,

$$\pi_0(c) = \mathrm{diag}(\alpha^n(c)) = \begin{bmatrix} \ddots & & & & \\ & \alpha^{-1}(c) & & & \\ & & \boxed{c} & & \\ & & & \alpha(c) & \\ & & & & \ddots \end{bmatrix},$$

$$U_0 = \begin{bmatrix} \ddots & & & & \\ \ddots & 0 & & & \\ & \mathbf{1}_{H_0} & \boxed{0} & & \\ & & \mathbf{1}_{H_0} & 0 & \\ & & & \mathbf{1}_{H_0} & \ddots \\ & & & & \ddots \end{bmatrix}.$$

Existence of covariant representations

We have

$$\pi_0(c)U_0 : e_n \otimes \xi \xrightarrow{U_0} e_{n+1} \otimes \xi \xrightarrow{\pi_0(c)} e_{n+1} \otimes \alpha^{n+1}(c)\xi$$

$$U_0\pi_0(\alpha(c)) : e_n \otimes \xi \xrightarrow{\pi_0(\alpha(c))} e_n \otimes \alpha^n(\alpha(c))\xi \xrightarrow{U_0} e_{n+1} \otimes \alpha^n(\alpha(c))\xi$$

hence

$$\pi_0(c)U_0 = U_0\pi_0(\alpha(c)), \quad \text{equivalently} \qquad \pi_0(\alpha(c)) = U_0^*\pi_0(c)U_0.$$

Existence of covariant representations

Proposition

The representation $U_0 \times \pi_0$ just constructed is injective on the covariance algebra $\mathscr{A}_0$.

Indeed, suppose $(U_0 \times \pi_0)\left(\sum_k u^k c_k\right) = 0$, i.e. $\sum_k U_0^k \pi_0(c_k) = 0$. Then for all $\xi, \eta \in H_0$ and all $m \in \mathbb{Z}$ we have

$$0 = \sum_{k=-\infty}^{\infty} U_0^k \pi_0(c_k)(e_0 \otimes \xi) = \sum_k U_0^k (e_0 \otimes \alpha^0(c_k)\xi) = \sum_k e_k \otimes \alpha^0(c_k)\xi$$

$$\text{and so} \quad 0 = \left\langle \sum_k e_k \otimes c_k \xi, e_m \otimes \eta \right\rangle = \langle c_m \xi, \eta \rangle_{H_0}$$

which shows that $c_m = 0$ and so, since m is arbitrary, that $\sum_k u^k c_k = 0$ in $\mathscr{A}_0$. □

Conclusion: Injective covariant representations exist!

The reduced C*-crossed product

Therefore the seminorm

$$\left\|\sum_k u^k c_k\right\|_r := \left\|(U_0 \times \pi_0)\left(\sum_k u^k c_k\right)\right\|_{\mathscr{B}(H)} = \left\|\sum_k U_0^k \pi_0(c_k)\right\|_{\mathscr{B}(H)}$$

is in fact a norm on $\mathscr{A}_0$, and is clearly an algebra seminorm satisfying the C*-condition. Therefore the completion of $\mathscr{A}_0$ in this norm is a C*-algebra.

Definition

The C*-reduced crossed product $\mathscr{C} \rtimes_{\alpha,r} \mathbb{Z}$ associated to the C*DS $(\mathscr{C}, \alpha)$ is the completion of $\mathscr{A}_0$ in the norm $\|\cdot\|_r$. Equivalently, it is the concrete C*-subalgebra of $\mathscr{B}(H)$ generated by $\pi_0(\mathscr{C})$ and U_0; it is the closure of $\mathscr{A}_0$ in the norm of $\mathscr{B}(H)$.