Operator Algebras: An introduction

18 October 2019

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$\mathscr{B}(\mathscr{H})$

Let $\mathscr{H}$ be a Hilbert space. The algebra of all bounded linear operators $T : \mathscr{H} \to \mathscr{H}$ is denoted $\mathscr{B}(\mathscr{H})$. It is complete under the norm

$$\|T\| = \sup\{\|Tx\| : x \in \mathrm{b}_1(\mathscr{H})\}$$

Moreover, it has an *involution* $T \to T^*$ defined via

$$\langle T^* x, y\rangle = \langle x, Ty\rangle \quad \text{for all } x, y \in \mathscr{H}.$$

This satisfies

$$\|T^* T\| = \|T\|^2 \qquad \text{the } C^* \text{ property.}$$

C*-algebras

Definition

(a) A Banach algebra $\mathscr{A}$ is a complex algebra equiped with a complete submultiplicative norm:

$$\|ab\| \leq \|a\| \, \|b\| \, .$$

(b) A C*-algebra $\mathscr{A}$ is a Banach algebra equiped with an involution[1] $a \to a^*$ and a complete submultiplicative norm (i.e. $\|ab\| \leq \|a\| \, \|b\|$) satisfying the C*-condition

$$\|a^* a\| = \|a\|^2 \qquad \text{for all } \; a \in \mathscr{A} \, .$$

A morphism $\phi : \mathscr{A} \to \mathscr{B}$ between C-algebras is a linear map that preserves products and the involution.*

Morphisms are automatically contractive, and 1-1 morphisms are isometric (algebra forces topology).

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[1] that is, a map on $\mathscr{A}$ such that $(a + \lambda b)^* = a^* + \bar{\lambda} b^*$, $(ab)^* = b^* a^*$, $a^{**} = a$ for all $a, b \in \mathscr{A}$ and $\lambda \in \mathbb{C}$

Basic Examples

- $\mathbb{C}$
- $C(K)$: K compact Hausdorff, $f^*(t) = \overline{f(t)}$: abelian, unital.
- $C_0(X)$: X locally compact Hausdorff, $f^*(t) = \overline{f(t)}$: abelian, nonunital (iff X non-compact).

Gelfand - Naimark Theorem I

All abelian C*-algebras can be represented as $C_0(X)$ for suitable X.

- $M_n(\mathbb{C})$: $A^* =$ conjugate transpose, $\|A\| = \sup\{\|Ax\|_2 : x \in \ell^2(n), \|x\|_2 = 1\}$: non-abelian, unital.
- $\mathscr{B}(\mathscr{H})$: non-abelian, unital.

Gelfand - Naimark Theorem II

All C*-algebras can be represented as closed selfadjoint subalgebras of $\mathscr{B}(\mathscr{H})$ for suitable $\mathscr{H}$.

Nonexamples:

- $A(\mathbb{D}) = \{f \in C(\overline{\mathbb{D}}) : f|_{\mathbb{D}} \text{ holomorphic}\}$ [2]
  A closed subalgebra of the C-algebra $C(\overline{\mathbb{D}})$ but not a *-subalgebra, because if $f \in A(\mathbb{D})$ then $\bar{f}$ is not holomorphic unless it is constant: $A(\mathbb{D}) \cap A(\mathbb{D})^* = \mathbb{C}\mathbf{1}$: antisymmetric algebra.*
- $T_n = \{(a_{ij}) \in M_n(\mathbb{C}) : a_{ij} = 0 \text{ for } i > j\}$ (upper triangular matrices).
  A closed subalgebra of the C-algebra $M_n(\mathbb{C})$ but not a *-subalgebra. Here $T_n \cap T_n^* = D_n$, the diagonal matrices: a maximal abelian selfadjoint algebra (masa) in M_n.*
- $M_{oo}(\mathbb{C})$: infinite matrices with finite support.
  To define norm (and operations), consider its elements as operators acting on $\ell^2(\mathbb{N})$ with its usual basis. This is a selfadjoint algebra, but not complete.
  Its completion is $\mathscr{K}$, the set of compact operators on ℓ^2: a non-unital, non-abelian C-algebra.*

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[2] $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$.

Matrix algebras

- If $\mathscr{A}$ is a C*-algebra and $n \in \mathbb{N}$, the space $M_n(\mathscr{A})$ of all matrices $[a_{ij}]$ with entries $a_{ij} \in \mathscr{A}$ becomes a *-algebra with product $[a_{ij}][b_{ij}] = [c_{ij}]$ where $c_{ij} = \sum_k a_{ik} b_{kj}$ and involution $[a_{ij}]^* = [d_{ij}]$ where $d_{ij} = d_{ji}^*$.

How to define a norm?

Special cases:

- Suppose $\mathscr{A}$ is $C_0(X)$; then norm $M_n(C_0(X))$ by identifying it (as a *-algebra) with $C_0(X, M_n)$, i.e. M_n-valued continuous functions on X vanishing at infinity.

- Suppose $\mathscr{A}$ is a C*-subalgebra of some $\mathscr{B}(\mathscr{H})$; then norm $M_n(\mathscr{A}) \subseteq M_n(\mathscr{B}(\mathscr{H}))$ by identifying $M_n(\mathscr{B}(\mathscr{H}))$ with $\mathscr{B}(\mathscr{H}^n)$.

General case: Use Gelfand - Naimark.

Παράδειγμα

Αναπαριστώ τον ℓ^∞ ως διαγώνιους τελεστές $H := \ell^2(\mathbb{Z}_+)$:

$$D(c) := \begin{bmatrix} c_0 & 0 & 0 & 0 & \dots \\ 0 & c_1 & 0 & 0 & \dots \\ 0 & 0 & c_2 & 0 & \dots \\ 0 & 0 & 0 & c_3 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}, \quad c \in \ell^\infty,$$

και θεωρώ τον

$$S = \begin{bmatrix} 0 & 0 & 0 & 0 & \dots \\ 1 & 0 & 0 & 0 & \dots \\ 0 & 1 & 0 & 0 & \dots \\ 0 & 0 & 1 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Ποιά είναι η [κλειστή] γραμμική θήκη ‘πολυωνύμων’ $\sum_{k=0}^{N} S^k D(c_k)$;

Ημι-σταυρωτά γινόμενα (semicrossed products)

Έστω δυναμικό σύστημα (K,σ) όπου K συμπαγής Hausdorff και $\sigma : K \to K$ συνεχής συνάρτηση. Θέτω $\mathscr{C} := C(K)$ και για κάθε $x \in K$ αναπαριστώ στον $H_x := \ell^2(\mathbb{Z}_+)$:

$$\pi_x(f) = \mathrm{diag}(f(\sigma^n(x))) = \begin{bmatrix} f(x) & 0 & 0 & 0 & \dots \\ 0 & f(x_1) & 0 & 0 & \dots \\ 0 & 0 & f(x_2) & 0 & \dots \\ 0 & 0 & 0 & f(x_3) & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix},$$

(όπου $f \in \mathscr{C}$, $x_n = \sigma^n(x)$)

$$S_x = \begin{bmatrix} 0 & 0 & 0 & 0 & \dots \\ 1 & 0 & 0 & 0 & \dots \\ 0 & 1 & 0 & 0 & \dots \\ 0 & 0 & 1 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Αθροίζω:

Ημι-σταυρωτά γινόμενα (semicrossed products)

$$\begin{aligned} \text{Ορίζω} \quad \pi(f) &:= \oplus_{x\in K}\pi_x(f) \\ S &:= \oplus_{x\in K} S_x \\ \text{στον} \quad H &:= \oplus_{x\in K} H_x. \end{aligned}$$

Το ημι-σταυρωτό γινόμενο $\mathscr{C} \rtimes_\sigma \mathbb{Z}_+$ είναι η κλειστή υπάλγεβρα (όχι *-υπάλγεβρα) της $\mathscr{B}(H)$ που παράγεται από τα $\{\pi(f) : f \in \mathscr{C}\} \cup \{S\}$.
Ελέγχεται ότι ικανοποιείται η 'covariance relation'

$$\pi(f)S = S\pi(f\circ\sigma)$$

(οπότε $S\pi(f)S\pi(g) = S^2\pi(f\circ\sigma)\pi(g) = S^2\pi((f\circ\sigma)g)$ κ.λπ.).
Έπεται ότι το $\mathscr{C} \rtimes_\sigma \mathbb{Z}_+$ είναι η κλειστή θήκη όλων των 'πολυωνύμων' $\sum_{n=0}^{N} S^n\pi(f_n)$ με συντελεστές $\pi(f_n)$ από την $\mathscr{C}$.

Σταυρωτά γινόμενα (crossed products)

Αν σ ομοιομορφισμός μπορώ να βάλω κάθε $H_x := \ell^2(\mathbb{Z})$, και $\pi_x(f) = \mathrm{diag}(f(\sigma^n(x)), n \in \mathbb{Z}$ και στη θέση του S_x το bilateral shift

$$\pi_x(f) = \begin{bmatrix} \ddots & & & & \\ & f(x_{-1}) & & & \\ & & \boxed{f(x)} & & \\ & & & f(x_1) & \\ & & & & \ddots \end{bmatrix} \quad S_x = \begin{bmatrix} \ddots & & & & & \\ \ddots & 0 & & & & \\ & 1 & \boxed{0} & & & \\ & & 1 & 0 & & \\ & & & 1 & \ddots & \\ & & & & \ddots & \end{bmatrix}.$$

Το σταυρωτό γινόμενο $\mathscr{C} \times_\sigma \mathbb{Z}$ είναι η κλειστή *-υπάλγεβρα της $\mathscr{B}(H)$ που παράγεται από τα $\{\pi(f) : f \in \mathscr{C}\} \cup \{S\}$. Είναι η κλειστή θήκη όλων των 'τριγωνομετρικών πολυωνύμων' $\sum_{n=-N}^{N} S^n \pi(f_n)$ με συντελεστές $\pi(f_n)$ από την $\mathscr{C}$.

Ένα συγκεκριμένο παράδειγμα

Έστω $K = \mathbb{T} = \{e^{it} : t \in [0, 2\pi]\}$ και $\sigma(e^{it}) = \lambda e^{it} = e^{i\theta} e^{it}$ όπου $\theta/2\pi$ άρρητος. Θέτουμε $H_2 = L^2(\mathbb{T}, \mu)$ (μέτρο Lebesgue).
Η αναπαράσταση π παράγεται απ' την εικόνα του $\pi(\zeta)$ (όπου $\zeta(e^{it}) = e^{it}$). Αφού εφαρμόσουμε μετασχηματισμό Fourier $\mathscr{F} : H_2 \to H$, ο $\pi(\zeta)$ αποδεικνύεται ότι αντιστοιχεί στον unitary τελεστή $V = M_\zeta$, δηλαδή:

$$(V\xi)(z) = z\xi(z) \quad \xi \in H_2, z = e^{it} \in \mathbb{T}.$$

Επίσης ο τελεστής S αντιστοιχεί στον unitary τελεστή U που ορίζεται από

$$(U\xi)(z) = \xi(\bar{\lambda} z).$$

Οι τελεστές αυτοί ικανοποιούν την covariance condition

$$VU = \lambda UV$$

($\sim$ η σχέση Weyl της Κβαντομηχανικής).