Fwd: Notes Hartz Lupini

Ηλιας Κατσουλης <makatsov@gmail.com>
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To: Katsoulis, Elias G <KATSOULISE@ecu.edu>

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Από: **Ηλιας Κατσουλης** <makatsov@gmail.com>
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Subject: Notes Hartz Lupini
To: <akatavol@math.uoa.gr>
Cc: <katsoulise@ecu.edu>

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Dilation theory in finite dimensions and matrix convexity.

M. Hartz and M. Lupini

What is a matrix convex set?

Let V be a complex vector space and

$\mathbb{X} = \{X_n\}_{n=1}^{\infty}$, where $X_n \subseteq M_n(V)$, $\forall\, n = 1, 2, \dots$

A matrix convex combination x from $\mathbb{X}$ is any expression of the form

$$x = \sum_{i=1}^{n} \gamma_i^* x_i \gamma_i$$

where $x_i \in M_{n_i}(V)$, $\gamma_i \in M_{n_i,n}(\mathbb{C})$, $\forall i$

and

$$\sum_{i=1}^{n} \gamma_i^* \gamma_i = I_n. \quad \text{(discuss convex hull)}$$

Examples of matrix convex combinations

(i) Any classical convex combination of elements of X_n is also a matrix convex combination

(ii) If $x \in X_n$ and $v \in M_{n,m}(\mathbb{C})$ any isometry then $v^* x v$ is a convex combination from $\mathbb{X}$

(iii) If $x_n \in M_n(V)$ and $x_m \in M_m(V)$, then

$$\begin{pmatrix} x_n & 0 \\ 0 & x_m \end{pmatrix}$$

is a convex combination from X.

Indeed

$$\begin{pmatrix} x_n & 0 \\ 0 & x_m \end{pmatrix} = (I_n, 0_{n,m})^* x_n (I_n, 0_{n,m}) + (0_{m,n}\ I_m)^* x_m (0_{m,n}, I_m)$$

with

$$(I_n, 0)^*(I_n, 0) + (0, I_m)^*(0, I_m) = \begin{pmatrix} I_n & 0 \\ 0 & I_m \end{pmatrix}$$

as required.

(iv) Any matricial convex combination of convex combinations from $\mathbb{X}$ is also a matricial convex combination from $\mathbb{X}$. (Excercise)

A (matrix) convex combination is said to be proper if each γ_i, $i=1,2,\dots,n$, is onto

A convex combination is trivial if $n_i = n$, $\forall i=1,2,\dots,n$, and each x_i is unitarily equivalent(?) to x

An element $x \in \underline{X}$ is said to be matrix extreme for $\underline{X}$ if any time x is expressed as a proper convex combination from $\underline{X}$, that combination has to be trivial.

We will be concerned with matrix convex sets $\underline{X}$, i.e., sets that equal their matrix convex hull.

If $\underline{X} = \{X_n\}_{n=1}^{\infty}$ with $X_n \subseteq M_n(V)$ and V admits a topology (always locally convex), then X_n is topologized using the product topology. We say that $\underline{X}$ is compact iff every X_n is compact.

<u>THEOREM 1</u> (matrix Minkowski) Let $\underline{X}$ be a a compact matrix convex set over a finite dimensional vector space V. Then $\underline{X}$ is the matrix convex hull of its matrix extreme points

There is also a matricial Krein–Millman !!!

Why should I care about matrix convex sets ???

The primary example of a matrix convex set is the matrix state space of an operator system, i.e, the collection of all unital, completely positive maps $\varphi : \mathcal{S} \longrightarrow M_n(\mathbb{C})$, $n \in \mathbb{N}$.

For a fixed $n \in \mathbb{N}$, such maps can be thought as a subset of $M_n(\mathcal{S}^*)$, and so by considering $V = \mathcal{S}^*$, we obtain that the collection of all matrix states (i.e. all ucp finite dimensional representations of $\mathcal{S}$) is a matrix convex set which is compact if we equip $\mathcal{S}^*$ with the w^*-topology.

A (not necessarilly unital) cp map $\varphi : \mathcal{S} \longrightarrow B(\mathcal{H})$ is said to be pure if given any cp map $\psi : \mathcal{S} \longrightarrow B(\mathcal{H})$ with $\psi \leq \varphi$, i.e. $\varphi - \psi$ completely positive, we have $\psi = \lambda \varphi$, $\lambda \in [0,1]$. (Arveson '69)

<u>THEOREM 2</u> (Arveson) Let $\mathcal{Q}$ be a unital C^*-algebra and let $\varphi: \mathcal{Q} \to B(\mathcal{H})$ be a cp map with minimal Stinespring dilation $(\pi, V, \mathcal{K})$. Then φ is pure if and only if $\pi: \mathcal{Q} \to B(\mathcal{K})$ is irreducible.

Proof (Sketch)

If $\pi: \mathcal{Q} \to B(\mathcal{K})$ is not irreducible, then consider a non-trivial projection $P \in \pi(\mathcal{Q})'$ and let

$$\psi : \mathcal{Q} \to B(\mathcal{H}); \ \mathcal{Q} \ni a \longrightarrow V^* \pi(a) P V$$

Then $\psi \leq \varphi$ but $\psi \neq \lambda\varphi$, $\lambda \in [0,1]$

Conversely assume that $\pi: \mathcal{Q} \to B(\mathcal{K})$ is irreducible and let $\hat{\varphi}: \mathcal{Q} \to B(\mathcal{H})$ cp with Stinespring repn $(\hat{\pi}, \hat{V}, \hat{\mathcal{K}})$ so that $\hat{\varphi} \leq \varphi$.

Consider the operator $(a_i a_j^*)_{ij} \geq 0$

$$\sum_{i=1}^{n} \pi(a_i) V \xi_i \xmapsto{\ T\ } \sum_{i=1}^{n} \hat{\pi}(a_i) \hat{V} \xi_i \qquad (*)$$

(This operator is a well-defined contraction because of $\hat{\varphi} \leq \varphi$ as c.p. maps.)

We now have

(i) $TV = \hat{V}$ (Set $n=1$, $a_1 = I$ in $(*)$)

(ii) $T\pi(a) = \hat{\pi}(a)T$, $\forall a \in \mathcal{Q}$. (Use (i) in $(*)$)

Therefore $T^*T\pi(a) = \pi(a)T^*T$, $\forall a \in \mathcal{Q} \xRightarrow[\pi]{\text{irreducible}} T^*T = \lambda I$, $\lambda \in [0,1]$

Hence

$$\hat{\varphi}(a) = \hat{V}^*\hat{\pi}(a)\hat{V} = V^*T^*\hat{\pi}(a)TV$$

$$\overset{(ii)}{=} V^*T^*T\pi(a)V = \lambda V^*\pi(a)V = \lambda\varphi(a). \qquad \square$$

THEOREM 3 (Arveson '69) Let $\mathcal{S}$ be an operator system inside a C^*-algebra $\mathcal{Q}$. Then any pure matrix state of $\mathcal{S}$ extends to a pure matrix state of $\mathcal{Q}$.

Let $\mathcal{S}$ be an operator system inside a C^*-algebra $\mathcal{Q}$ and $\varphi : \mathcal{S} \longrightarrow B(\mathcal{H})$, $\dim \mathcal{H} < \infty$, ucp map.

We would like to know under what assumptions φ dilates to a finite dimensional representation of $\mathcal{Q}$, i.e. there exists representation

$$\pi : \mathcal{Q} \longrightarrow B(K), \quad \mathcal{H} \subseteq K, \quad \dim K < \infty$$

so that $\varphi(s) = P_{\mathcal{H}} \pi(s)|_{\mathcal{H}}$, $\forall s \in S$.

Note that Arveson's Extension Theorem and Stinespring's dilation Theorem guarantee that such a π exists without the extra condition $\dim K < \infty$.

A C^*-algebra $\mathcal{Q}$ is called FDI (Courtney and Shulman, 2017) if all irreducible representations of $\mathcal{Q}$ are finite dimensional

Abelian C^*-algebras are FDI

PROPOSITION 4 Let $\mathcal{Q}$ be an FDI C^*-algebra, $S \subseteq \mathcal{Q}$ an operator system and $\varphi : S \longrightarrow B(\mathcal{H})$, $\dim \mathcal{H} < \infty$, a ucp map. Then TFAE

(i) φ dilates to a finite dimensional repn of $\mathcal{Q}$

(ii) φ is a matrix convex combination of restrictions of pure matrix states of $\mathcal{Q}$

Proof

(i) $\Rightarrow$ (ii) Assume that φ dilates to $\pi : \mathcal{Q} \longrightarrow B(K)$, $\dim K < \omega$ so that $\varphi = V^*(\pi|_S)V$. So it is enough to show that $\pi|_S$ is the desired convex combination.

Since $\dim K < \omega$, $\pi \cong \oplus \pi_i$, where all

π_i are irreducible repns of $\mathcal{Q}$, hence pure matrix states of $\mathcal{Q}$. Hence

$$\pi|_S \cong \oplus(\pi_i|_S) = \text{matrix convex combination of restrictions of pure matrix states of } \mathcal{Q}.$$

(ii) $\Rightarrow$ (i)

Assume that

$$\varphi = \sum_j \gamma_j^* \varphi_j \gamma_j, \quad \sum \gamma_j^* \gamma_j = I \quad (*)$$

and each φ_j is a restriction of a pure matrix state

Hence, $\varphi_j(a) = V_j^* \sigma_j(a) V_j$, $a \in \mathcal{Q}$, and each $\sigma_j : \mathcal{Q} \longrightarrow B(K_j)$ is irreducible. Since $\mathcal{Q}$ is FDI, $\dim K_j < \omega$.

Hence $\varphi(s) = \sum \gamma_j^* V_j^* \sigma_j(s) V_j \gamma_j$, $s \in S$

Set $\sigma = \oplus_j \sigma_j$ and

$$\psi : \oplus_j B(K_j) \longrightarrow B(\mathcal{H}); \quad (b_j)_j \longmapsto \sum \gamma_j^* V_j^* b_j V_j \gamma_j$$

and so $\varphi(s) = (\psi \circ \sigma)(s)$, $s \in S$ $\quad (**)$

Because of $(*)$, ψ is unital and also CP. Hence it admits a Stinespring dilation

$$\begin{tikzcd} & \mathcal{B}(\mathcal{L}) \arrow[d, "V^* \cdot V"] \\ \bigoplus_j \mathcal{B}(\mathcal{K}_j) \arrow[ur, "\tau"] \arrow[r, "\psi"'] & \mathcal{B}(\mathcal{H}) \end{tikzcd}$$

Because $\bigoplus_j \mathcal{B}(\mathcal{K}_j)$ and $\mathcal{B}(\mathcal{H})$ are finite dimensional the proof of Stinespring's Thm implies $\dim \mathcal{L} < \infty$.

Revisiting $(**)$

$$\varphi(s) = \psi(\sigma(s)) = V^* \tau(\sigma(s)) V$$

$$= V^* (\tau \circ \sigma)(s) V$$

Since $\tau \circ \sigma$ is finite dimensional rep. of $\mathcal{A}$, we are done. $\square$

We will be concerned with $\dim \mathcal{S} < \infty$, so in light of the matricial Minkowski Thm we saw earlier it makes sense to characterize the matrix extreme points of an operator system $\mathcal{S}$.

<u>THEOREM 5</u> (Farenick, 2000 JLMS) Let $\mathcal{S}$ be an

operator system in a C^*-algebra $\mathcal{A}$.
Then a matrix state φ of S is a matrix extreme point of the matricial state space of S if and only if φ is pure.

For S a C^*-algebra this gives a very concrete picture of the matrix extreme points of the matricial state space.

<u>THEOREM 6</u> Let S be a finite dimensional operator system inside an FDI C^*-algebra. Let

$$\varphi : S \longrightarrow B(\mathcal{H}), \quad \dim \mathcal{H} < \infty$$

be a ucp map. Then φ dilates to a finite dimensional representation of $\mathcal{A}$.

<u>Proof</u>. By the matricial Minkowski Thm, φ is a convex combination of matrix extreme points $\{\varphi_j\}_j$ for the matricial space of S, i.e.,

$$\varphi = \sum_j \gamma_j^* \varphi_j \gamma_j$$

By Farenick's Theorem, the φ_j are pure ucp maps and so by Theorem 3 (Arveson), the φ_j extend to pure ucp maps ψ_j of $\mathcal{A}$. Hence

$$\varphi = \sum_j \gamma_j^* (\psi_j | s) \gamma_j$$

and the conclusion follows from Proposition 4. $\square$

So what is the big deal about this result???

Any contraction $T \in B(\mathcal{H})$ admits a unitary dilation $U_T \in B(\mathcal{H}_T)$
(This actually establishes the existence of a ucp map $\varphi_T : C(\mathbb{T}) \to B(\mathcal{H})$; $C(\mathbb{T}) \ni p(z) \longmapsto p(T)$.)

HOWEVER, even if $\dim \mathcal{H} < \infty$, $\dim \mathcal{H}_T = \infty$, except in the case where T is already a unitary.
How can we remedy this???

<u>THEOREM 7</u> (Egervary, 1954) Let $T \in B(\mathcal{H})$, $\dim \mathcal{H} < \infty$, be a contraction and let $N \in \mathbb{N}$. Then there exists a finite dimensional Hilbert space $K \supseteq \mathcal{H}$ and a unitary operator $U \in B(K)$ so that for any polynomial p of degree at most N,

$$p(T) = P_{\mathcal{H}} \, p(U)|_{\mathcal{H}} .$$

Proof Consider the finite dimensional system

$$\mathcal{S} = \{z^{-N}, \dots, 1, z, \dots z^{N}\} \subseteq C(\mathbb{T})$$

and the ucp map $\varphi_T : C(\mathbb{T}) \to B(\mathcal{H})$; $p(z) \longrightarrow p(T)$ discussed above

By Theorem 6, φ_T dilates to a finite dimensional representation of $C(\mathbb{T})$, i.e., we have

$$\pi : C(\mathbb{T}) \longrightarrow B(\mathcal{K}), \ \dim \mathcal{K} < \infty, \ *\text{-repn}$$

so that

$$\varphi_T(p) = P_K \pi(p)\big|_K, \quad p \in \mathcal{S} \text{ polynomial}$$

If $U = \pi(z)$, then U is unitary and $\pi(p) = p(U)$ $\square$

We can provide a more contemporary application

Recall that if T_1, T_2 are commuting contractions, then by Ando's Theorem, they admit a joint unitary dilation, i.e., there exists Hilbert space $\mathcal{H}_T \supseteq \mathcal{H}$ and commuting unitaries $U_1, U_2 \in B(\mathcal{H}_T)$ so that

$$P_{\mathcal{H}} U_1^m U_2^n \big|_{\mathcal{H}} = T_1^m T_2^n, \quad m, n \geq 0$$

(and so

$$P_{\mathcal{H}} (U_1^*)^m (U_2^*)^n \big|_{\mathcal{H}} = T_1^{*m} T_2^{*n}, \quad m, n \geq 0)$$

Therefore as before we have a ucp map

$$C(\mathbb{T}^2) \supseteq \mathcal{P}(z_1, z_2) + \overline{\mathcal{P}(z_1, z_2)} \xrightarrow{\varphi_T} B(\mathcal{H}); \quad p(z_1, z_2) \to p(T_1, T_2)$$

where $\mathcal{P}(z_1, z_2)$ denotes the analytic polynomials in z_1, z_2

THEOREM 8 (McCarthy and Shalit, 2013) Let T_1, T_2 be commuting contractions and let $N \in \mathbb{N}$. Then there exists finite dimensional Hilbert space $\mathcal{K} \supseteq \mathcal{H}$ and commuting unitaries U_1, U_2 so that for any analytic polynomial of degree N we have:

$$p(T_1, T_2) = P_{\mathcal{H}}\, p(U_1, U_2)\big|_{\mathcal{H}}$$

Proof. As before consider the ucp map

$$\varphi_T : \mathcal{P}(z_1, z_2) + \overline{\mathcal{P}(z_1, z_2)} \longrightarrow B(\mathcal{H}) : p(z_1, z_2) \to p(T_1, T_2)$$

and by Theorem 6 obtain a dilation of φ to a finite dimensional representation of $C(\mathbb{T}^2)$. $\square$

The proof of the matricial Minkowski Thm

We want to show that for a fixed compact matricial convex set set $\mathbb{X} = \{X_n\}_{n \in \mathbb{N}}$, $X_n \subseteq M_n(V)$, $\dim V < \infty$ every point in $\mathbb{X}$ is a matricial convex combination of extreme points

The key tool: For every $n \in \mathbb{N}$ we define (tr = normalized trace)

$$\Gamma_n(X) := \{ (\gamma^*\gamma, \gamma^* x \gamma) \mid \gamma \in M_{k,n}(\mathbb{C}),\ tr(\gamma^*\gamma) = 1,\ x \in X_k,\ k \in \mathbb{N} \}$$

The set Γ_n connects matricial convexity with the familiar convexity !!!

LEMMA 9 $\Gamma_n(X)$ is convex

Proof

Proof

Let $(\gamma^*\gamma, \gamma^* x \gamma) \in \Gamma_n(X)$. Sine $ran(\gamma)$ has dimension less than n, say $r \leq n$, let $v \in M_{n,r}(\mathbb{C})$ isometry $vv^* = ran(\gamma)$.

Then

$$\gamma^* x \gamma = \gamma^* v v^* x v v^* \gamma = (v^*\gamma)^* \underbrace{v^* x v}_{\in X_r} (v^*\gamma)$$

and $tr((v^*\gamma)^* v^*\gamma) = tr(\gamma^* v \cdot v^* \gamma) = tr(\gamma^*\gamma) = 1$

Putting Lemma 9 and 10 together

COROLLARY 11 $\Gamma_n(X)$ is a compact convex subset of a finite dimensional vector space. (Hence the classical result of Minkowski applies)

The following makes the important connection.

LEMMA 12 Let $(\gamma^*\gamma, \gamma^*x\gamma) \in \Gamma_n(X)$, with $x \in X_k$ and $\gamma \in M_{k,n}(\mathbb{C})$ surjective. If $(\gamma^*\gamma, \gamma^*x\gamma)$ is an extreme point for $\Gamma_n(X)$, then x is an extreme point for $\overline{X}$.

Proof

Assume that

$$x = \sum \gamma_i^* x_i \gamma_i \quad , \quad x_i \in X_{k_i}$$

is a proper convex combination. (and so γ_i surjective)

Then

$$\gamma^* x \gamma = \sum (\gamma_i\gamma)^* x_i (\gamma_i\gamma)$$

and

$$\gamma^*\gamma = \gamma^*\left(\sum \gamma_i^*\gamma_i\right)\gamma = \sum (\gamma_i\gamma)^* \gamma_i\gamma$$

therefore

$$(\gamma^*\gamma, \gamma^* x \gamma) = \sum_i \left((\gamma_i\gamma)^*\gamma_i\gamma , (\gamma_i\gamma)^* x_i \gamma_i\gamma \right)$$

$$= \sum_i t_i \left(\underline{\left(\frac{\gamma_i\gamma}{t_i^{1/2}}\right)^* \frac{\gamma_i\gamma}{t_i^{1/2}}} , \left(\frac{\gamma_i\gamma}{t_i^{1/2}}\right)^* x_i \frac{\gamma_i\gamma}{t_i^{1/2}} \right) \qquad (*)$$

since we want

$$tr\left(\frac{(\gamma_i\gamma)^*}{t_i^{1/2}} \frac{\gamma_i\gamma}{t_i^{1/2}} \right) = 1 \qquad \text{for membership in } \Gamma_n(x),$$

we obtain

$$tr(\gamma^*\gamma_i^*\gamma_i\gamma) =: t_i$$

Furthermore

$$\sum_i t_i = tr\left(\sum \gamma^*\gamma_i^*\gamma_i\gamma\right) = tr\left(\gamma^*\underbrace{\left(\sum_i \gamma_i^*\gamma_i\right)}_{= I}\gamma\right) = t(\gamma^*\gamma) = 1$$

So $(*)$ is a true convex combination in $\Gamma_n(x)$

Since $(\gamma^*\gamma, \gamma^* x\gamma)$ is extreme, we obtain from the first entries that

$$\frac{\gamma^*\gamma_i^*\gamma_i\gamma}{t_i} = \gamma^*\gamma \quad \text{or}$$

$$\gamma^*(\gamma_i^*\gamma_i - t_i I_k)\gamma = 0$$

Since γ is onto, $\gamma^*(\gamma_i^*\gamma_i - t_i I_n) = 0$ and by taking adjoints $\gamma_i^*\gamma_i - t_i I_n = 0$

Hence $\gamma_i / t_i^{1/2}$ is unitary (and $k_i = k$, $\forall i$).

From the second entries in $(*)$ we obtain

$$\gamma^* x \gamma = \frac{(\gamma_i \gamma)^*}{t_i^{1/2}} x_i \frac{\gamma_i \gamma}{t_i^{1/2}} \qquad \gamma^*\left(\frac{\gamma_i}{t_i}\right)^* x_i \left(\frac{\gamma_i}{t_i}\right)\gamma^*$$

or

$$\gamma^*\left(x - \left(\frac{\gamma_i}{t_i^{1/2}}\right)^* x_i \left(\frac{\gamma_i}{t_i^{1/2}}\right)\right)\gamma = 0$$

Arguing as earlier

$$x = \left(\frac{\gamma_i}{t_i}\right)^* x_i \left(\frac{\gamma_i}{t_i}\right) \quad \text{i.e.,} \quad x_i \text{ unit. equiv. to } x.$$

$\square$

We can now prove the matricial Minkowski

If $x \in X_k$, then $(I, x) \in \Gamma_k(x) =$ convex, compact subset of f.d space

Hence

$$(I, x) = \Sigma\, t_i \left(\gamma_i^*\gamma_i \,,\, \gamma_i^* x_i \gamma_i\right) \quad \text{extreme in } \overline{X}$$

and so

$$x = \sum \underbrace{(t_i^{1/2}\gamma_i)^*}_{\hat{\gamma}_i} x_i \underbrace{(t_i^{1/2}\gamma_i)}_{\hat{\gamma}_i}$$

Furthermore

$$\sum_i (t_i^{1/2}\gamma_i)^*(t^{1/2}\gamma_i) = \sum t_i \gamma_i^* \gamma_i = I$$

from the first entries ...

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