From Synchronous Games to the Connes Embedding Problem

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joint work with many people

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Overview

In the paper MIP*=RE, Ji, Natarajan, Vidick, Wright and Yuen, prove that Connes' Embedding Problem has a negative answer.
Their proof actually shows that Tsirelson's Problem is false, which by work of Junge et al, Fritz and Ozawa has been shown to be equivalent to Kirchberg's Problem, which by work of Kirchberg is equivalent to Connes' Embedding Problem.
Their paper is 160+ pages and the other results to get from their work to the Connes' embedding problem via this route is also quite daunting.
Their proof uses some deep results from complexity theory, Turing machines, and a compression algorithm to prove the existence of a synchronous game with certain properties.
In this talk we outline a somewhat shorter route, given the existence of this game, based on work with several coauthors.

We don't claim that this is a better route, because it leaves out a lot of very nice results, just shorter. At the same time it introduces some different results.
The remaining challenge to finding a proof that can be digested in a few settings is to find another way to prove the existence of the game implicitly constructed in MIP*=RE.
Our hope is that perhaps a more robust theory of synchronous games could clarify some of these constructions.
I feel that the theory of synchronous games and their algebras has a place in the field of operator algebras alongside of the algebras affiliated with other objects, like groups, goupoids, quantum groups, graphs, etc.

From CEP to TP

Connes' Embedding Problem(CEP): Does every separable II_1-factor $(\mathcal{M}, tr)$ embed in a trace preserving manner into an ultrapower $(\mathcal{R}^\omega, \tau_\omega)$ of the hyperfinite II_1-factor $(\mathcal{R}, \tau)$.
Kirchberg's Problem(KP): Is there a unique C*-norm on $C^*(\mathbb{F}_\infty) \otimes C^*(\mathbb{F}_\infty)$, i.e., is $C^*(\mathbb{F}_\infty) \otimes_{min} C^*(\mathbb{F}_\infty) = C^*(\mathbb{F}_\infty) \otimes_{max} C^*(\mathbb{F}_\infty)$?
In 1993, Kirchberg proved that CEP and KP are equivalent.
Tsirelson's Problem(s): Do several different models for the conditional probability densities produced by entangled quantum measurements coincide?(Definitions later)
In 2010, M. Junge, M. Navascues, C. Palazuelos, D. Perez-Garcia, V. B. Scholz, R. F. Werner and separately, T. Fritz, proved that KP true implied that two of Tsirelson's models coincided. Later, Ozawa proved the converse.

Whether or not these two particular models give rise to the same sets of densities has come to be known as **Tsirelson's Problem(TP)**.
Thus, it was known that $CEP \iff KP \iff TP$.
In January of 2020, in the paper MIP*=RE, Ji, Natarajan, Vidick, Wright and Yuen, prove that TP is false, and hence, all three are false.
Our route: 1) $syncTP \iff CEP$(bypassing KP)
2) Their synchronous game shows syncTP is false. In fact, we will show why this could be potentially easier to show.
3) Every synchronous game has an affiliated *-algebra and it is this algebra of their game that fails to be embeddable in Connes' sense.

Quantum Correlations: Tsirelson and Connes

Suppose that Alice and Bob each have n quantum experiments and each experiment has k outcomes. We let $p(a,b|x,y)$ denote the conditional probability that Alice gets outcome a and Bob gets outcome b given that they perform experiments x and y respectively. If we assume that the labs are separate but that they share an entangled state, then there are several possible models for describing the set of all such tuples.

One model is that Alice and Bob have finite dimensional state spaces $\mathcal{H}_A$ and $\mathcal{H}_B$. For each experiment x, Alice has projections $\{E_{x,a}, 1 \le a \le k\}$ such that $\sum_a E_{x,a} = I_A$. Similarly, for each y, Bob has projections $\{F_{y,b} : 1 \le b \le k\}$ such that $\sum_b F_{y,b} = I_B$. They share an entangled state(i.e., a unit vector) $\psi \in \mathcal{H}_A \otimes \mathcal{H}_B$ and

$$p(a,b|x,y) = \langle E_{x,a} \otimes F_{y,b} \psi | \psi \rangle.$$

We let $C_q(n,k) = \{p(a,b|x,y) : \text{ obtained as above } \} \subseteq \mathbb{R}^{n^2m^2}$.
We let $C_{qc}(n,k)$ denote the possibly larger set that we could obtain if instead of requiring the common state space to be a tensor product, we just required one common state space $\mathcal{H}$, with state $\psi \in \mathcal{H}$ and demanded that $E_{a,x}F_{y,b} = F_{y,b}E_{xa,x}$ for all a,b,x,y, i.e., a commuting model.
Tsirelson was the first to examine these sets and study the relations between them. In fact, he wondered if they could be equal.
We now know that $C_q(n,k)$ is not closed(Slofstra, Dykema-P-Prakash), while $C_{qc}(n,k)$ is closed.
Let $C_{qa}(n,k)$ denote the closure of $C_q(n,k)$.

Tsirelson's Problem(TP): Is $C_{qa}(n,k) = C_{qc}(n,k), \forall n,k$?

Theorem (JNPPSW, Ozawa)

$C_{qa}(n,k) = C_{qc}(n,k), \forall n,k$ if and only if $C^*(\mathbb{F}_\infty) \otimes_{min} C^*(\mathbb{F}_\infty) = C^*(\mathbb{F}_\infty) \otimes_{max} C^*(\mathbb{F}_\infty)$, i.e., $TP \iff KP$.

Synchronous Correlations

A correlation $p(a, b|x, y)$ is called *synchronous* provided $p(a, b|x, x) = 0, \ \forall a \neq b, \forall x.$ We write $C_q^s(n, k), C_{qa}^s(n, k), C_{qc}^s(n, k)$ for the subset of synchronous correlations.

Theorem (P-Severini-Stahkle-Todorov-Winter)

$p(a, b|x, y) \in C_{qc}^s(n, k)$ iff there exists a C-algebra $\mathcal{A}$ generated by projections $\{E_{x,a} : 1 \leq x \leq n, 1 \leq a \leq k\}$ satisfying $\sum_{a=1}^k E_{x,a} = I, \forall x$ and a trace $\tau : \mathcal{A} \to \mathbb{C}$ such that*

$$p(a, b|x, y) = \tau(E_{x,a}E_{y,b}).$$

Moreover, $p(a, b|x, y) \in C_q^s(n, k)$ iff $\mathcal{A}$ can be taken to be finite dimensional.

Idea of proof: Given $p(a, b|x, y) = \langle E_{x,a} F_{y,b} \psi | \psi \rangle$ one uses Cauchy-Schwarz and the synchronous condition to show that $E_{x,a}\psi = F_{x,a}\psi$.
Hence, $p(a, b|x, y) = \langle E_{x,a} E_{y,b} \psi | \psi \rangle$.
Now one shows that the state on $\mathcal{A} = C^*(\{E_{x,a} : 1 \leq x \leq n, 1 \leq a \leq k\})$ induced by ψ is tracial.
This didn't give a characterization of $C_{qa}^s(n, k)$.

Theorem (Kim-P-Schafhauser)

Fix integers $n, k \geq 1$. *For* $\big(p(i,j|v,w)\big) \in \mathbb{R}^{n^2k^2}$, *the following are equivalent:*

1. $\big(p(a,b|x,y)\big) \in C_{qa}^s(n,k)$;
2. *(synchronous approximation) there are synchronous correlations* $\big(p_m(a,b|x,y)\big) \in C_q^s(n,k)$ *with*

$$p_m(a,b|x,y) \to p(a,b|x,y) \quad \forall\ a,b,x,y;$$

3. *there exist projections* $E_{x,a} \in \mathcal{R}^\omega, \sum_{a=1}^k E_{x,a} = I$ *such that*

$$p(a,b|x,y) = \tau_\omega(E_{x,a}E_{y,b}).$$

Proof uses results from the theory of amenable traces, especially Kirchberg's work.

Theorem (Dykema-P(syncTP))

CEP has an affirmative answer if and only if $C^s_{qa}(n,k) = C^s_{qc}(n,k),\ \forall n,k$, *i.e., CEP* $\iff$ *syncTP.*

A proof of this theorem does not need JNPPSW, Ozawa or KP. But does use the equivalence of 1) and 2), along with Pisier's "linearization trick" to show that syncTP is equivalent to the "matricial microstates conjecture" directly. This microstates conjecture is one of the more immediate equivalences of CEP. This proof is a remark in DP, because DP came before KPS, and we remark that if only we knew the equivalence of 1) and 2) then our proof gives the above theorem, without using these other results.

Finite Input-Output Games(Non-local Games)

These are games where two *cooperating* but *non-communicating* players, Alice and Bob try to give *correct* answers to questions posed by the Referee.

For each *round* of the game, the cooperating players each receive an input, i.e., a question, from the Referee from some finite set of inputs I_A, I_B.

They must each produce an output, i.e., an answer, belonging to some finite set O_A, O_B.

The game $\mathcal{G}$ has *rules* given by a function

$$\lambda : I_A \times I_B \times O_A \times O_B \to \{0, 1\}$$

where $\lambda(x, y, a, b) = 1$ means that if Alice and Bob receive inputs x, y, respectively and produce respective outputs a, b, then they win. If $\lambda(x, y, a, b) = 0$, they lose.

They both know the rule function and can create a strategy for winning, but once the game starts Alice and Bob must produce their outputs without knowing what input the other received and without knowing what output the other produced. This is what is meant by *non-communicating*.

A *random strategy* is identified with a conditional probability density $p(a, b|x, y)$.

A random strategy is called *perfect* if it has 0 probability of producing a losing output, i.e.,

$$\lambda(x, y, a, b) = 0 \implies p(a, b|x, y) = 0.$$

Synchronous games

A game is called synchronous if $I_A = I_B$, $O_A = O_B$ and the rules include the condition that whenever the players receive the same input(question) they must produce the same output(answer). If $p(a, b|x, y)$ represents the probability that when receiving inputs x, y the players produce outputs a, b, respectively, then to be a perfect strategy it must satisfy,

$$\forall x, p(a, b|x, x) = 0 \text{ whenever } a \neq b$$

i.e., be a synchronous correlation.

The *-algebra of a synchronous game

Let $\mathcal{G} = (I, O, \lambda)$ be a synchronous game. By the *-algebra of the game, $\mathcal{A}(\mathcal{G})$, we mean the "universal" *-algebra generated by projections $\{e_{x,a} : x \in I, a \in O\}$ satisfying:

- $\forall x \in I, \sum_{a \in O} e_{x,a} = I,$
- $\lambda(a, b, x, y) = 0 \implies e_{x,a} e_{y,b} = 0$

Theorem (Helton-Meyer-P-Satriano, KPS)

Let $\mathcal{G}$ be a synchronous game then:

- *$\mathcal{G}$ has a perfect deterministic strategy iff there is a unital *-homomorphism of $\mathcal{A}(\mathcal{G})$ into $\mathbb{C}$,*
- *$\mathcal{G}$ has a perfect q-strategy iff there is a unital *-homomorphism of $\mathcal{A}(\mathcal{G})$ into M_p for some p,*
- *$\mathcal{G}$ has a perfect qc-strategy iff there is a unital *-homomorphism of $\mathcal{A}(\mathcal{G})$ into a tracial C*-algebra.*
- *$\mathcal{G}$ has a perfect qa-strategy iff there is a unital *-homomorphism of $\mathcal{A}(\mathcal{G})$ into $\mathcal{R}^\omega$.*

To prove this theorem one takes the perfect strategy $p(a, b|x, y)$, expresses it as a $\tau(E_{x,a}E_{y,b})$ for some C*-algebra and note that the elements $E_{x,a}$ will satisfy the relations used to define $\mathcal{A}(\mathcal{G})$.

MIP*=RE proves the existence of a synchronous game with a perfect qc-strategy but no perfect qa-strategy. Hence syncTP is false.
Bonus: By our theory the *-algebra of this game has a trace but that algebra cannot embed into $\mathcal{R}^\omega$.
So one only needs to make the description of this game more explicit to have a concrete algebra violating CEP.

Values and Synchronous Values of Games

MIP*=RE uses the concept of the value of a game to get their result. The synchronous theory could make this part simpler. Given a game $\mathcal{G}$, a distribution on the inputs $\pi(x, y)$ and a probabilistic strategy $p(a, b|x, y)$ the expected probability of winning the game is:

$$val(\mathcal{G}, \pi, p) := \sum_{x,y,a,b} \pi(x, y)p(a, b|x, y)\lambda(x, y, a, b)$$

If we insist that $\pi(x, y) > 0, \forall x, y$, then $val(\mathcal{G}, \pi, p) = 1 \iff p$ is a perfect strategy.

For $t = q, qa, qc$ we set

$$\omega_t(\mathcal{G}, \pi) := sup\{val(\mathcal{G}, \pi, p) : p \in C_t\}.$$

Note that since the closure of C_q is C_{qa} and $C_{qa} \subseteq C_{qc}$ we have that

$$\omega_q(\mathcal{G}, \pi) = \omega_{qa}(\mathcal{G}, \pi) \le \omega_{qc}(\mathcal{G}, \pi).$$

MIP*=RE construct a synchronous game such that

$$\omega_q(\mathcal{G}, \pi) < 1/2 < \omega_{qc}(\mathcal{G}, \pi) = 1$$

and this separation of values gives their results.
Similarly, we set

$$\omega_t^s(\mathcal{G}, \pi) := sup\{val(\mathcal{G}, \pi, p) : p \in C_t^s\}.$$

Then

$$\omega_q^s(\mathcal{G}, \pi) = \omega_{qa}^s(\mathcal{G}, \pi) \le \omega_{qc}^s(\mathcal{G}, \pi),$$

and

$$\omega_t^s(\mathcal{G}, \pi) \le \omega_t(\mathcal{G}, \pi).$$

Moreover, for synchronous games one can easily see that

$$\omega_{qc}(\mathcal{G}, \pi) = 1 \iff \omega_{qc}^{s}(\mathcal{G}, \pi) = 1.$$

Thus, for the game in MIP*=RE we have that

$$\omega_{q}^{s}(\mathcal{G}, \pi) \leq \omega_{q}(\mathcal{G}, \pi) < 1/2 < \omega_{qc}^{s}(\mathcal{G}, \pi).$$

So it is potentially easier to show that

$$\omega_{q}^{s}(\mathcal{G}, \pi) < 1/2 < \omega_{qc}^{s}(\mathcal{G}, \pi)$$

for their game and this could yield some shortening of their proof.

$E\upsilon\chi\alpha\rho\iota\sigma\tau\omega$

References

KPS: A synchronous game for binary constraint systems(with S.-J. Kim and C. Schafhauser)
HMPS: Algebras, synchronous games and chromatic numbers of graphs(with J.W. Helton, K.P. Meyer, and M. Satriano)
PSSTW: Estimating Quantum Chromatic Numbers(with S. Severini, D. Stahlke, I. Todorov and A. Winter)
DPP: Non-closure of the set of quantum correlations via graphs(with K. Dykema and J. Prakash)