TRANSITION BETWEEN WAVE FUNCTIONS AND STATES (ctd)

Reference ($3.3 "Alice and Bob meet Banach: The interface of asymptotic geometric analysis and quantum information theory")

In the context of chapter 3 of "Alice and Bob meet Banach: The interface of asymptotic geometric analysis and quantum information theory" ,we describe a physical system with a wave function $\psi \in H \otimes E$ where H is the Hilbert space that corresponds to the "world" we can perceive and measure and E the Hilbert space corresponding to the "environment" we have no access to. The probability that an "observable" quantity U will be in the eigenstate $u_j \otimes e_k$ (where $\{u_j\}$ and $\{e_k\}$ are orthonormal bases of H and E respectively) will be $|< \psi \mid u_j \otimes e_k >|^2$.

As we can only perform measurements in H we assume that there is a related quantity U_H which acts on H and we are interested in the probability that this quantity will be in the eigenstate u_j.

We are interested in the corresponding to ψ state in H , that will give the same probability of measurement of U_H in u_j as U in all the states $\{u_j \otimes e_k\}_{k=1...}$, which is of course $\sum_{k=1...} |< \psi \mid u_j \otimes e_k >|^2$. This state is defined in the book as the ***H*-marginal of ψ** . In the case when $\psi = \xi \otimes \eta$ it is easy to see that the H -marginal of ψ is the expected η.

In the general case we write ψ as :

$\psi = \sum_{i=1}^{r} a_i . \xi_i \otimes \eta_i$, the Schmidt decomposition of ψ and we look for the H-marginal of ψ .

We pick the orthonormal basis $\{\boldsymbol{\eta_k}\}$ for $\boldsymbol{E}$ and we denote by p_j the probability of measurement of U_H in u_j as U in all the states $\{u_j \otimes \eta_k\}_{k=1...}$ we find that :

$p_j = \sum_{k=1...} |< \psi \mid u_j \otimes \eta_k >|^2 =$

$=\sum_{k=1\ldots}\sum_{i=1}^{r}\sum_{l=1}^{r} a_i < \xi_i \otimes \eta_i \mid u_j \otimes \eta_k >$

$. a_l . < u_j \otimes \eta_k \mid \xi_l \otimes \eta_l >$

$=\sum_{k=1\ldots}\sum_{i=1}^{r}\sum_{l=1}^{r} a_i < \xi_i \mid u_j >. \delta_{ik} . a_l . < u_j \mid \xi_l >. \delta_{kl}$

$= \sum_i \mid a_i \mid^2 . \mid < \xi_i \mid u_j > \mid^2$

$=$ Tr $(\rho_H . \mid u_j > < u_j \mid)$

with $\rho_H = \sum_i \mid a_i \mid^2 . \mid \xi_i > < \xi_i \mid$

so that, as the book states, the "obvious" candidate for the H-marginal of ψ, namely $\sum_i a_i \xi_i$ will not do and we need to introduce the operator formalism as a replacement to the wave-function formalism. A vector ψ is identified with an operator

$O_\psi : H \otimes E \rightarrow H \otimes E$ with

$O_\psi(\varphi) = < \varphi \mid \psi > . \psi$. It is now a relatively easy matter to find the **H-marginal of O_ψ** $= \sum_i \mid a_i \mid^2 . \mid \xi_i > < \xi_i \mid$ as long as the relevant probabilities "when we are in a quantum state ρ_H (operator)" of obtaining a measurement corresponding to $\mid u_j >$ are given by

$p_j =$ Tr $(\rho_H . \mid u_j > < u_j \mid)$

(the trace is taken in H).

In this way of looking at probabilities we can reformulate the problem as follows: When we are in (general case) quantum state ρ ∈ D($H \otimes E$) can we find the H–marginal $\rho_H \in$ D(H) s.t.

$\forall \mid u_j > \in H$

Tr $(\rho_H . \mid u_j > < u_j \mid) =$ Tr $[\rho . (\mid u_j > < u_j \mid \otimes 1_E)]$

(where the first trace is on H and the second on $H \otimes E$)

<u>Partial Trace</u>

Define a map Tr_E : B($H \otimes E$) → B(H)

by $Tr_E(\mathrm{A} \otimes \mathrm{B}) = \mathrm{Tr}(\mathrm{B}) \,.\, \mathrm{A}$ and extend by linearity

(need to check that it is well defined on the tensor product)

Lemma 1

For $\rho = | \psi > < \psi |$ with $\psi = \sum_{i=1}^{r} a_i . \xi_i \otimes \eta_i$, the Schmidt decomposition of ψ , we have that $Tr_E(\rho) = \rho_H$

Proof: $Tr_E(\rho) = Tr_E \sum_{i=1}^{r} \sum_{l=1}^{r} a_i\, a_l \,.\, | \xi_i \otimes \eta_i > < \xi_l \otimes \eta_l |$

$= \sum_{i=1}^{r} \sum_{l=1}^{r} a_i\, a_l \,.\, Tr_E\, | \xi_i > < \xi_l \otimes | \eta_i >< \eta_l |$

$= \sum_{i=1}^{r} \sum_{l=1}^{r} a_i\, a_l \,.\, Tr(| \eta_i >< \eta_l |)\, (| \xi_i > < \xi_l) |$

$= \sum_{i=1}^{r} \sum_{l=1}^{r} a_i\, a_l \,.\, Tr(| \xi_i > < \xi_l |) \,.\, \delta_{il}$

$= \sum_i | a_i |^2 . \; | \xi_i > < \xi_i |$

$= \rho_H$

Lemma 2 For $\rho = | \psi > < \psi |$ with $\psi = \sum_{i=1}^{\dim H} \sum_{j=1}^{dimE} a_{ij} . \xi_i \otimes \eta_j$

the decomposition of ψ in the orthonormal basis $\{ \xi_i \otimes \eta_j \}$

we have that $Tr_E(\rho) = \sum_{j=1}^{\dim E} | w_j > < w_j |$

where $| w_j > = \sum_{i=1}^{\dim H} a_{ij} \; | \xi_i >$. Furthermore $Tr_E(\rho) \in \mathrm{D}(H)$.

Proof: Using the linearity of Tr_E (assumed checked)

$Tr_E(\rho) = \sum_{i=1}^{dimH} \sum_{j=1}^{dimE} \sum_{k=1}^{dimH} \sum_{l=1}^{dimE} a_{ij} \,.\, \overline{a_{kl}} \,.\, | \xi_i \otimes \eta_j > < \xi_k \otimes \eta_l |$

and noting that

$Tr_E\, (| \xi_i \otimes \eta_j > < \xi_k \otimes \eta_l |) = | \xi_i > < \xi_k | \,.\, \delta_{jl}$

we get

$Tr_E(\rho) = \sum_{i=1}^{dimH} \sum_{j=1}^{dimE} \sum_{k=1}^{dimH} a_{ij} \,.\, \overline{a_{kj}} \; | \xi_i > < \xi_k |$

or $\boldsymbol{Tr_E(\rho) = \sum_{j=1}^{\dim E} | w_j > < w_j |}$

(as $< w_j | = \sum_{k=1}^{\dim H} \overline{a_{kj}} \,.\, < \xi_k |$)

The fact that $Tr_E(\rho)$ is positive semidefinite is immediate from the fact that it is a sum of positive operators and

$$\mathrm{Tr}\ \ (Tr_E(\rho)) = \sum_{i=1}^{dimH} \sum_{j=1}^{dimE} \sum_{k=1}^{dimH} \ a_{ij} \,.\, \overline{a_{kJ}}\ \mathrm{Tr}(\mid \xi_i > < \xi_k \mid)$$

$$= \sum_{i=1}^{dimH} \sum_{j=1}^{dimE} \sum_{k=1}^{dimH} \ a_{ij} \,.\, \overline{a_{kJ}}\ \delta_{ik} =$$

$$= \sum_{i=1}^{dimH} \sum_{j=1}^{dimE} \mid a_{ij} \mid^2$$

$=\ 1$ as ψ is unitary.