Poisson, without walks

Introduction Let[1] G be a locally compact Hausdorff topological group[2] and μ a Borel probability measure on G.

A function $f : G \to \mathbb{C}$ is said to be *μ-harmonic* if it satisfies the equation

$$f(t) = \int_G f(ts)d\mu(s)\,.$$

In other words, if it is a *fixed point* of the map $P : f \to Pf$ where

$$(Pf)(t) = \int_G f(ts)d\mu(s)\,.$$

Considering this map to be defined on $L^\infty(G)$, we observe that it is a *linear, contractive, positive (i.e. $f \geq 0 \Rightarrow Pf \geq 0$) and unital map* (i.e. $P\mathbf{1} = \mathbf{1}$):

$$P : L^\infty(G) \to L^\infty(G).$$

Furthermore, it is continuous with respect to the weak* topology that $L^\infty(G)$ has as the Banach space dual of $L^1(G)$.

The space of bounded harmonic functions is the set of fixed points of this map:

$$H_\mu^\infty := \{f \in L^\infty(G) : Pf = f\}.$$

It is therefore a linear space, containing the unit of $L^\infty(G)$, which is closed under complex conjugation ($f \in H_\mu^\infty \Rightarrow \bar{f} \in H_\mu^\infty$) and closed in the weak* topology. However, it is *not closed under (pointwise) products.*

Claim *H_μ^∞ is the range of a contractive, unital projection $E : L^\infty(G) \to L^\infty(G)$.*

The idea is the following: For each $n \in \mathbb{Z}_+$, the map $P^n : L^\infty(G) \to L^\infty(G)$ (where $P^0 = I$, $P^2 = P \circ P, \dots$) leaves the linear space H_μ^∞ (elementwise) fixed, hence so do the averages

$$E_n := \frac{1}{n+1}(I + P + P^2 + \cdots + P^n).$$

Now it is a fact that the space $\mathcal{B}(L^\infty(G)) := \{T : L^\infty(G) \to L^\infty(G) : \text{linear, continuous}\}$ has a certain linear space topology $\mathcal{T}$ in which its unit ball is compact. [3] A compactness argument (Markov-Kakutani, [1, Theorem 10.1]) shows that if K is the $\mathcal{T}$-closed convex hull of $\{I, P, \dots P^n, \dots\}$ and $E \in \bigcap_n E_n(K)$, then $EP = PE = E$.

This map $E : L^\infty(G) \to L^\infty(G)$ is also linear, contractive, positive and unital (but not necessarily weak* continuous). Now since PE=E, each Ef in the range of E is μ-harmonic. And conversely, if $f \in H_\mu^\infty$, then f is fixed by P, hence by every convex combination of powers of P, hence by E, which is a $\mathcal{T}$-limit of such combinations. Thus,

$$H_\mu^\infty = E(L^\infty(G)).$$

[1] nowalk, January 3, 2017, revised January 10, 2017

[2] possibly it is enough to assume G is a Hausdorff topological semigroup

[3] $\mathcal{T}$ is the weak* topology of $\mathcal{B}(L^\infty(G))$ as the dual of a certain Banach space.

Now we define a *new product* $f \times g$ on H^∞_μ by

$$f \times g = E(fg), \quad f, g \in H^\infty_\mu.$$

It is a non-trivial fact (special case of the Choi-Effros theorem see later) that this product is associative and satisfies the C*-property, $\|f^* \times f\|_\infty = \|f\|_\infty$.

Since the new product is obviously commutative, the structure $(H^\infty_\mu, *, \times, \|\cdot\|_\infty)$ is an abelian C*-algebra; since moreover $(H^\infty_\mu, \|\cdot\|_\infty)$ is a dual Banach space, it follows from Sakai's Theorem that this C*-algebra is in fact a von Neumann algebra.

But an abelian Neumann algebra is in fact isometrically *-isomorphic to $L^\infty(\Omega, \nu)$ for an appropriate measure space (Ω, ν).

We conclude that there exists a measure space (Ω, ν) and a linear onto isometry

$$f \to \hat{f} : H^\infty_\mu \to L^\infty(\Omega, \nu)$$

which send constants to constants, nonnegative functions to nonnegative functions and satisfies $\widehat{f \times g} = \hat{f} \cdot \hat{g}$.

Continuous harmonic functions [4]

A bounded function $f : G \to \mathbb{C}$ is continuous if $\lim_{s \to e} |f(sx) - f(x)| = 0$ for every $x \in G$. We say that f is *left uniformly continuous (luc)* if $\lim_{s \to e} |f(sx) - f(x)| = 0$ uniformly in $x \in G$, i.e. if $\lim_{s \to e} \|L_s f - f\|_\infty = 0$ where $L_s f(x) = f(sx)$. We denote by $\mathcal{A}$ the algebra $C^b_{luc}(G)$ of bounded left uniformly continuous functions $f : G \to \mathbb{C}$.

For a Borel probability measure μ on G, write

$$H_\mu := \{f \in \mathcal{A} : Pf = f\} \subseteq H^\infty_\mu$$

A topological space X is a *G-space* if there exists a continuous map

$$G \times X \to X : (s, \xi) \to s \cdot \xi$$

which is (jointly) continuous and satisfies $s \cdot (t \cdot \xi) = (st) \cdot \xi$ and $e \cdot \xi = \xi$. We write $G \curvearrowright X$.

The action of G on X induces a map $s \to L_s$ [5] of G to operators on $C^b(X)$ given by

$$(L_s f)(\xi) := f(s \cdot \xi).$$

Note that each L_s is a linear isometry, which is onto because $L_{s^{-1}} L_s = I$. Also, it is easy to check that

$$L_{st} = L_t L_s \quad \text{for all } \ s, t \in G.$$

The main result we wish to prove is the following

Theorem 1 *There exists a compact Hausdorff G-space Π_μ and a linear unital onto isometry*

$$T : C(\Pi_\mu) \to H_\mu$$

which is equivariant, *i.e. satisfies $T \circ L_s = L_s \circ T$ for all $s \in G$.*

[4] Our approach is based on [3]

[5] this is not an action: the map $s \to L_s$ reverses products; usually one defines $(\lambda_s f)(t) = f(s^{-1}t)$, and this does give an action.

In fact, we will also prove that T 'comes from' integration against a suitable measure:

Proposition 2 *There exists a Borel probability measure ν on the space Π_μ so that T is given by*

$$T(\hat{f})(s) = \int_{\Pi_\mu} \hat{f}(s \cdot \xi) d\nu(\xi) \quad \text{for all } \hat{f} \in C(\Pi_\mu),\ s \in G.$$

Proof See later.

This is the *Poisson formula*, which expresses every harmonic function $f = T(\hat{f})$ on G as the integral of a function $\hat{f}$ defined on the Poisson boundary of (G, μ).

Proposition 3 (Uniqueness) *The space Π_μ is essentially unique, in the following sense:*

If K is a compact Hausdorff G-space and $T' : C(\Pi_\mu) \to H_\mu$ is a linear onto equivariant isometry, then there exists a homeomorphism $\phi : K \to \Pi_\mu$ making the actions $G \curvearrowright K$ and $G \curvearrowright \Pi_\mu$ conjugate, that is

$$s \cdot \xi = s \cdot \phi(\xi) \qquad \text{for all } \xi \in K,\ s \in G.$$

Proof Consider the composition

$$\Phi : C(\Pi_\mu) \xrightarrow{T} H_\mu \xrightarrow{T'} C(K).$$

This is a unital onto isometry. The classical Banach-Stone Theorem [1, Theorem VI.2.1] states that Φ must be of the from $\Phi(f) = f \circ \phi$ where $\phi : K \to \Pi_\mu$ is a homeomorphism.

Now for all $\xi \in K$, $f \in C(\Pi_\mu)$ and $s \in G$ we have, by the definitions of ϕ and L_s,

$$(L_s\Phi(f))(\xi) = \Phi(f)(s \cdot \xi) = f(\phi(s \cdot \xi))$$
$$(\Phi(L_s f))(\xi) = (L_s f)(\phi(\xi)) = f(s \cdot \phi(\xi))$$

But, since T and T' are equivariant, Φ must be equivariant, i.e. $L_s(\Phi(f)) = \Phi(L_s(f))$; therefore

$$f(\phi(s \cdot \xi)) = f(s \cdot \phi(\xi)) \quad \text{for all } f \in C(\Pi_\mu),$$
$$\text{hence} \quad \phi(s \cdot \xi) = s \cdot \phi(\xi)$$

since continuous functions separate points of compact (Hausdorff) spaces. □

The proof of Theorem 1 will consist in two main steps: first we will show that H_μ is the range of a contractive, positive, unital projection E, and then we will use this to construct a different product on H_μ which will give it the structure of an abelian unital C*-algebra; now Gelfand theory shows that this C*-algebra must be of the form $C(\Pi_\mu)$ for a certain compact Hausdorff space Π_μ.

Proposition 4 *H_μ is the range of a contractive, unital projection $E : C^b_{luc}(G) \to C^b_{luc}(G)$ mapping non-negative functions to non-negative functions (hence real-valued functions to real-valued functions).*

Proof First note that the map P given by

$$(Pf)(t) = \int_G f(ts) d\mu(s)$$

maps the algebra $\mathcal{A} := C^b_{luc}(G)$ into itself.

Indeed, we have

Claim 1 *If $f \in \mathcal{A}$, then $L_t f \in \mathcal{A}$ for all $t \in G$.*

Proof If $x \in G$, since $L_{t^{-1}}$ is an isometry,

$$\begin{aligned}\|L_x(L_t f) - L_t f\|_\infty &= \|L_{t^{-1}}(L_x(L_t f) - L_t f)\|_\infty \\ &= \|(L_{t^{-1}} L_x L_t)(f) - f)\|_\infty = \|L_{txt^{-1}}(f) - f)\|_\infty \to 0\end{aligned}$$

as $x \to e$, because $txt^{-1} \to e$. □

Claim 2 *If $f \in \mathcal{A}$, then $Pf \in \mathcal{A}$.*

Indeed,

$$\begin{aligned}|L_x(Pf)(t) - (Pf)(t)| &\leq \int |(L_{xt} f)(s) - (L_t f)(s)| d\mu(s) \\ &\leq \|L_{xt} f - L_t f\|_\infty = \|L_t L_x f - L_t f\|_\infty = \|L_t(L_x f - f)\|_\infty = \|L_x f - f\|_\infty\end{aligned}$$

for all $t \in G$ and so $\|L_x(Pf) - Pf\|_\infty \to 0$ as $x \to e$.

As in the case of H^∞_μ, the averages (E_n) of (P^n) leave H_μ invariant. The idea is to transfer the action from $\mathcal{A}$ to its dual, where the weak* compactness of the unit ball will allow the use of fixed point techniques.

Since every $f \in \mathcal{A}$ is continuous, hence Borel, and bounded, the measure μ induces a continuous linear form $\hat{\mu} \in \mathcal{A}^*$ by

$$\langle \hat{\mu}, f \rangle := \int_G f d\mu \qquad (f \in \mathcal{A}).$$

We write the defining formula for P in the form

$$(Pf)(t) = \langle \hat{\mu}, L_t f \rangle \ .$$

More generally, for all $\alpha \in \mathcal{A}^*$ we may form $\langle \alpha, L_t f \rangle$ (since $L_t f \in \mathcal{A}$) and this gives a function of t, which we denote by $\alpha \cdot f$, that is

$$(\alpha \cdot f)(t) := \langle \alpha, L_t f \rangle \, .$$

Claim 3 *If $f \in \mathcal{A}$, then $\alpha \cdot f \in \mathcal{A}$ for all $\alpha \in \mathcal{A}^*$.*

Indeed,

$$\begin{aligned}|L_x(\alpha \cdot f)(t) - (\alpha \cdot f)(t)| &\leq |\langle \alpha, L_{xt} f - L_t f \rangle| \leq \|\alpha\| \, \|L_{xt} f - L_t f\|_\infty \\ &= \|\alpha\| \, \|L_t L_x f - L_t f\|_\infty = \|\alpha\| \, \|L_x f - f\|_\infty\end{aligned}$$

for all $t \in G$ and so $\|L_x(\alpha \cdot f) - \alpha \cdot f\|_\infty \to 0$ as $x \to e$.

In particular,

$$(Pf)(t) = \langle \hat{\mu}, L_t f \rangle = (\hat{\mu} \cdot f)(t) \, .$$

Claim 4 *For any $f \in \mathcal{A}$, $\alpha \in \mathcal{A}^*$ and $s \in G$ we have*

$$\alpha \cdot (L_s f) = L_s(\alpha \cdot f) \, . \tag{1}$$

Indeed, since $L_t(L_s)f = L_{st}f$,

$$\begin{aligned}(\alpha \cdot (L_s f))(t) &= \langle \alpha, L_t(L_s)f \rangle = \langle \alpha, L_{st}f \rangle \\ L_s(\alpha \cdot f)(t) &= (\alpha \cdot f)(st) = \langle \alpha, L_{st}f \rangle \ .\end{aligned}$$

Recall the definition of convolution of two measures on G:

$$\int f d(\mu * \nu) := \iint f(ts) d\mu(t) d\nu(s), \qquad f \in C_c(G).$$

When $f \in \mathcal{A}$, we may write this in the form

$$\begin{aligned}\langle \hat{\mu} * \hat{\nu}, f \rangle &= \iint (L_t f)(s) d\mu(t) d\nu(s) = \int \left(\int (L_t f)(s) d\nu(s) \right) d\mu(t) \\ &= \int \langle \hat{\nu}, L_t f \rangle \, d\mu(t) = \int (\hat{\nu} \cdot f)(t) d\mu(t) \\ &= \langle \hat{\mu}, \hat{\nu} \cdot f \rangle \ .\end{aligned}$$

It is therefore natural to define, for $\alpha, \beta \in \mathcal{A}^*$, a 'convolution' $\alpha * \beta$ by:

$$\langle \alpha * \beta, f \rangle = \langle \alpha, \beta \cdot f \rangle \qquad f \in \mathcal{A}. \tag{2}$$

Note that

$$(\alpha * \beta) \cdot f = \alpha \cdot (\beta \cdot f) \qquad f \in \mathcal{A}\,. \tag{3}$$

Indeed,

$$((\alpha * \beta) \cdot f)(s) = \langle \alpha * \beta, L_s f \rangle = \langle \alpha, \beta \cdot (L_s f) \rangle \overset{(1)}{=} \langle \alpha, L_s(\beta \cdot f) \rangle = (\alpha \cdot (\beta \cdot f))(s)\,.$$

This implies associativity of $*$:

$$(\alpha * \beta) * \gamma = \alpha * (\beta * \gamma). \tag{4}$$

Indeed, for all $f \in \mathcal{A}$,

$$\langle (\alpha * \beta) * \gamma, f \rangle = \langle \alpha * \beta, \gamma \cdot f \rangle = \langle \alpha, \beta \cdot (\gamma \cdot f) \rangle \overset{(3)}{=} \langle \alpha, (\beta * \gamma) \cdot f) \rangle = \langle \alpha * (\beta * \gamma), f \rangle \ .$$

It is immediate from the definition that the linear map $\mathcal{A}^* \to \mathcal{A}^* : \alpha \to \alpha * \beta$ is weak*-continuous for all $\beta \in \mathcal{A}^*$.

Claim 5 *If $\nu \in M(G)$, the map $\mathcal{A}^* \to \mathcal{A}^* : \alpha \to \hat{\nu} * \alpha$ is weak-* continuous.*

Proof Let $f \in \mathcal{A}$. By the definition of $\mathcal{A}$, the map $s \to L_s f : G \to \mathcal{A}$ is $\|\cdot\|_\infty$-continuous and therefore ν-integrable; thus there exists an element of $\mathcal{A}$, which we denote $f \cdot \hat{\nu}$, defined by

$$f \cdot \hat{\nu} := \int_G L_s f d\nu(s)$$

and of course, for every $\alpha \in \mathcal{A}^*$, since $\int_G L_s f d\nu(s)$ is a norm limit of linear combinations of translates of f,

$$\langle a, f \cdot \hat{\nu} \rangle = \int_G \langle \alpha, L_s f \rangle \, d\nu(s)$$

Thus,

$$\langle a, f \cdot \hat{\nu} \rangle = \int_G (\alpha \cdot f)(s) d\nu(s) = \langle \hat{\nu}, \alpha \cdot f \rangle \overset{(2)}{=} \langle \hat{\nu} * \alpha, f \rangle \ .$$

Therefore, if $\alpha_i \to \alpha$ in the weak*-topology, then for all $f \in \mathcal{A}$,

$$\langle \hat{\nu} * \alpha_i, f \rangle = \langle a_i, f \cdot \hat{\nu} \rangle \to \langle a, f \cdot \hat{\nu} \rangle = \langle \hat{\nu} * \alpha, f \rangle$$

which proves the Claim.

Now let $L_0 \subseteq \mathcal{A}^*$ be the convex hull of $\{\hat{\mu}, \hat{\mu}^2, \hat{\mu}^3, \dots\}$ and let L be its weak* closure: a compact convex subset of the unit ball of $\mathcal{A}^*$.

Observe that $\hat{\mu}$ is a *state* on $\mathcal{A}$, that is, a linear functional that takes nonnegative functions to nonnegative functions and the constant function $\mathbf{1}$ to 1. The same is true for $\hat{\mu}^2, \hat{\mu}^3, \dots$ (since $\langle \hat{\mu}^2, f \rangle = \langle \hat{\mu}, \hat{\mu} \cdot f \rangle$ etc.) and therefore (since the set of states is convex and weak* closed in $\mathcal{A}^*$) for any element of the set L.

Write S for the linear map

$$\mathcal{A}^* \to \mathcal{A}^* : \alpha \to \hat{\mu} * \alpha.$$

By Claim 5, S is weak* continuous. Since $S(L_0) \subseteq L_0$, it follows that $S(L) \subseteq L$.

By the Markov-Kakutani theorem [1, Theorem 10.1], S has a fixed point, say $\beta \in L$.

Observe that β is idempotent, i.e. $\beta * \beta = \beta$. Indeed β is a weak*-limit of a net (β_i) of convex combinations of elements $\hat{\mu}^n$ of L_0. But $\hat{\mu} * \beta = S(\beta) = \beta$; furthermore,

$$(\hat{\mu} * \hat{\mu}) * \beta \overset{(4)}{=} \hat{\mu} * (\hat{\mu} * \beta) = \hat{\mu} * \beta = \beta$$

and, inductively, $(\hat{\mu}^n) * \beta = \beta$ for all $n \in \mathbb{N}$. By linearity, $\beta_i * \beta = \beta$ for all i and so $\beta * \beta = \lim_i(\beta_i * \beta) = \beta$ by weak* continuity of the map $\alpha \to \alpha * \beta$ (Claim 5).

Conclusion of the proof We claim that the map

$$E : \mathcal{A} \to \mathcal{A} : f \to \beta \cdot f$$

satisfies the requirements of the proposition.
First, since β is in the unit ball of $\mathcal{A}^*$, we have

$$|(Ef)(t)| = |\langle \beta, L_t f \rangle| \leq \|\beta\| \, \|L_t f\| \leq \|L_t f\| = \|f\| \quad \text{for all } t,$$

hence $\|Ef\| \leq \|f\|$.
Secondly, since β is a state, $(E\mathbf{1})(t) = \langle \beta, L_t \mathbf{1} \rangle = \langle \beta, \mathbf{1} \rangle = 1$ for all t, hence $E\mathbf{1} = \mathbf{1}$. Also, if $f \geq 0$ then $L_t f \geq 0$ and so $\langle \beta, L_t f \rangle \geq 0$, i.e. $(Ef)(t) \geq 0$ for all t; thus $Ef \geq 0$.
Thirdly, $E \circ E = E$. Indeed, for all $f \in \mathcal{A}$, since β is idempotent, if $\alpha \in \mathcal{A}^*$ we have $(\alpha * \beta) * \beta = \alpha * (\beta * \beta) = \alpha * \beta$ and so

$$\langle \alpha, E(E(f)) \rangle = \langle \alpha, \beta \cdot (\beta \cdot f) \rangle = \langle (\alpha * \beta) * \beta, f) \rangle = \langle (\alpha * \beta, f) \rangle = \langle (\alpha, \beta \cdot f) \rangle = \langle \alpha, E(f) \rangle \ .$$

Finally, we claim that $E(\mathcal{A}) = H_\mu$.

Proof If $f \in E(\mathcal{A})$, writing $f = E(g) = \beta \cdot g$ for some $g \in \mathcal{A}$, we have

$$\begin{aligned}\hat{\mu} \cdot f =& \hat{\mu} \cdot (\beta \cdot g) \overset{(4)}{=} (\hat{\mu} * \beta) \cdot g \\ &= \beta \cdot g = f.\end{aligned}$$

This shows that $\hat{\mu} \cdot f = f$ so that f is harmonic.

Conversely suppose that $\hat{\mu} \cdot f = f$. Let $\beta = \lim_i \beta_i$ where $\beta_i \in L_0$. Since $\hat{\mu}^n \cdot f = f$ for all n, we have $f = \beta_i \cdot f$ for every i. Now for all $\nu \in M(G)$, Claim 5 shows that $\hat{\nu} * \beta = \lim \hat{\nu} * \beta_i$ in the weak* topology of $\mathcal{A}^*$, and therefore

$$\langle \nu, \beta \cdot f \rangle = \langle \nu * \beta, f \rangle = \lim \langle \hat{\nu} * \beta_i, f \rangle = \lim \langle \hat{\nu}, \beta_i \cdot f \rangle = \langle \nu, f \rangle \ .$$

In particular, for all $t \in G$, setting $\nu = \delta_t$ we obtain

$$(\beta \cdot f)(t) = \langle \delta_t, \beta \cdot f \rangle = \langle \delta_t, f \rangle = f(t)$$

and so $\beta \cdot f = f$. □

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