Injective G-spaces

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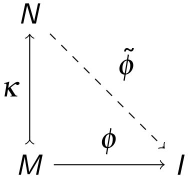
Injectivity

Definition 1

An object I of a category $\mathscr{C}$ is called **injective** when every morphism into it extends, i.e. given objects $M \subseteq N$ and a morphism $\phi : M \to I$ there is a morphism $\tilde{\phi} : N \to I$ such that $\tilde{\phi}|_M = \phi$.



Write $\quad I \in \mathrm{Inj}(\mathscr{C})$.

Categories

$E \subseteq \mathscr{B}(H_1)$, $F \subseteq \mathscr{B}(H_2)$.

$$E \xrightarrow{\phi} F \quad \rightsquigarrow \quad M_n(E) \xrightarrow{\phi_n} M_n(F) : \phi_n([a_{ij}]) = [\phi_n(a_{ij})].$$

1. $\mathfrak{O}$: operator spaces with completely bounded maps. $(\sup_n \|\phi_n\| < \infty)$
2. $\mathfrak{O}_1$: operator spaces with complete contractions. $(\forall n, \|\phi_n\| \leq 1)$
3. $\mathfrak{S}$: operator systems (selfadjoint, unital) with completely positive maps. $(\forall n, \phi_n \geq 0)$
4. $\mathfrak{S}_1$: operator systems with *unital* completely positive (ucp) maps.

Examples

In the category of Banach spaces with linear contractions, $\mathbb{C}$ is injective (Hahn-Banach).
Also $\ell^\infty(\Gamma)$ is injective for any Γ.

In the category $\mathfrak{S}$ of operator systems with completely positive maps, $\mathscr{B}(H)$ is injective (Arveson).

G-(operator) systems or G-spaces

Fix a group G.
Let $E \subseteq \mathscr{B}(H_1)$, $F \subseteq \mathscr{B}(H_2)$ be operator systems s.t. $G \curvearrowright E$ and $G \curvearrowright F$ by ucp maps. A G-**morphism or** G-**map** $E \xrightarrow{\phi} F$ is a ucp map s.t.

$$s \cdot \phi(x) = \phi(s \cdot x) \quad \text{for all } x \in E, s \in G.$$

Let $G\mathfrak{S}_1$: operator systems with unital completely positive (ucp) *G-maps*.

In the category $G\mathfrak{S}_1$, the space $\ell^\infty(G)$ with $(s \cdot x)(t) := x(s^{-1}t)$ for $x \in \ell^\infty(G)$ is G-injective (Hamana).

More generally, if $V \in \mathrm{Inj}(\mathfrak{S}_1)$ then $\ell^\infty(G, V) \in \mathrm{Inj}(\mathrm{G}\mathfrak{S}_1)$ with $(s \cdot x)(t) := x(s^{-1}t)$ for $x \in \ell^\infty(G, V)$.
In particular, $\ell^\infty(G, \mathscr{B}(H)) \in \mathrm{Inj}(\mathrm{G}\mathfrak{S}_1)$.
On the other hand, $\mathbb{C}$ with the trivial G-action $(s \cdot \lambda := \lambda)$, is G-injective iff G is amenable!

Injective envelopes

Definition 2

Given a G-space F we say that (E,κ) is **an injective envelope of** F, provided that

i) E is injective in $G\mathfrak{S}_1$,

ii) $\kappa:\ F\to E$ is a 1-1 ucp G-map,

iii) if E_1 is G-injective with $\kappa(F)\subseteq E_1\subseteq E$, then $E_1=E$.

So (E,κ) is a 'minimal' G-injective extension of E.

Existence of a G-Injective envelope (M. Hamana)

Let $F \subseteq \mathscr{B}(H)$ be a G-space. Embed F into $W := \ell^\infty(G, \mathscr{B}(H))$ by $j : F \to W$ where $j(x)(s) := s^{-1} \cdot x, x \in F, s \in G$ (note j is a G-map).

A map $\varphi : W \to W$ is **an** F**-map** if φ is a ucp G-map and $\varphi(x) = x$ for all x in F.

An F-map φ such that $\varphi \circ \varphi = \varphi$ is called an F-**projection**. Thus, an F-projection φ is a ucp projection onto $E = \varphi(W)$, with $F \subseteq E$, but we do not demand $F = E$.

We define a partial order on F-projections by setting $\psi \prec \varphi$ provided that $\psi \circ \varphi = \psi = \varphi \circ \psi$.

Given an F-map φ, we define an F-**seminorm** $p_\varphi = \| \cdot \|_\varphi$ on W by setting $p_\phi(x) = \|x\|_\phi = \|\varphi(x)\|$.

Existence of a G-Injective envelope (M. Hamana)

Let $W \in \mathrm{Inj}(\mathrm{G}\mathfrak{S}_1)$.

Proposition 3

If $\phi : W \to W$ is an idempotent G-map, then $\phi(W) \in \mathrm{Inj}(\mathrm{G}\mathfrak{S}_1)$.

NB. For the proof, embed $W \hookrightarrow \ell^\infty(G, \mathscr{B}(H))$ and use a compactness argument.

Proposition 4

Let $F \subseteq W$ be a G-space. Then there exist minimal F-seminorms on W.

Theorem 5

Let $F \subseteq W$ be a G-space. If $\varphi : W \to W$ is an F-map such that p_φ is a minimal F-seminorm, then φ is a minimal F-projection and the range $\varphi(W)$ of φ is a G-injective envelope of F.