V a countable state space.

$P = (p(x,y))_{x,y\in V}$ a trasition matrix.

$V^{\mathbb{N}_0}$ is endowed with the usual Borel σ-field $\mathcal{B}$, which is the minimal σ-fiele containing all elementary cylinders $\{y_0\}\times\dots\times\{y_k\}\times V^{\{k+1,k+2,\dots\}}$, $y_0,\dots,y_k\in V$, $k\in\mathbb{N}_0$.

S will denote the left shift on $V^{\mathbb{N}_0}$, for which $S(x_0,x_1,\dots) = (x_1,x_2,\dots)$.

P_x, for $x\in V$, denotes the law of the Markov chain $X_0, X_1,\dots$ with transition matrix P and initial distribution $\delta_{\{x\}}$, i.e., $P_x(X_0 = x) = 1$.

$P_\theta := \sum_{x\in V}\theta(x)P_x$ is the law of the Markov chain with initial distribution θ.

$\mathcal{T} := \bigcap_{n=0}^{\infty} S^{-n}\mathcal{B}$ is the tail σ-field. A tail function is a function $f: V^{\mathbb{N}_0}\to\mathbb{R}$ measurable with respect to $\mathcal{T}$.

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Let V, P be given as above. An automorphism of (V,P) is a permutation γ of V s.t. $p(\gamma(x),\gamma(y)) = p(x,y)$ $\forall x,y\in V$.

A Markov chain with transition matrix P and state space V is called transitive or spatially homogeneous, if $\mathrm{Aut}(V,P)$ acts transitively on V; that is for any $z,w\in V$ there exists $\gamma\in\mathrm{Aut}(V,P)$ s.t. $\gamma(z)=w$.

Examples: Simple random walk on a transitive graph is a transitive MC.
Convolution random walks on groups are transitive MC.

[A graph (V,E) is transitive if, for any $x,y\in V$, there exists an automorphism φ of the graph with $\varphi(x)=y$, an automorphism of the graph (V,E) being a bijection $\varphi: V\to V$ s.t. $[\varphi(x),\varphi(y)]\in E$ iff $[x,y]\in E$.]

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If X is a discrete r.v. its entropy is defined as

$$H(X) = -\sum_x P(X=x)\ln P(X=x)$$

More generally, given a σ-field $\mathcal{F}$, one defines conditional entropy as

$$H(X\mid\mathcal{F}) = -\sum_x P(X=x\mid\mathcal{F})\ln P(X=x\mid\mathcal{F}).$$

LEMMA: Let X be a discrete r.v. on a probability space $(\Omega,\mathcal{F},P)$.

(i) If $\mathcal{G}\subseteq\mathcal{F}$, then $H(X\mid\mathcal{G})\geq H(X\mid\mathcal{F})$.
Equality iff $P(X=\cdot\mid\mathcal{F}) = P(X=\cdot\mid\mathcal{G})$ a.s.

(ii) If $\mathcal{G}_n\supseteq\mathcal{G}_{n+1}$, $n\in\mathbb{N}$, is a decreasing sequence of σ-fields, $\mathcal{G}_n\subseteq\mathcal{F}$ $\forall n\in\mathbb{N}$, with $\mathcal{G}_n\downarrow\mathcal{G}$ then $H(X\mid\mathcal{G}_n)\uparrow H(X\mid\mathcal{G})$.

Consider a transitive chain $(X_n)_{n\in\mathbb{N}_0}$ on a countable V with $X_0 = o$. Transitivity gives the condition $H(X_{n+m}\mid X_m) = H(X_n)$ and hence $H(X_{n+m})\leq H(X_m)+H(X_n)$. (*)
In particular, $H(X_1)<\infty \Rightarrow H(X_n)<\infty$ $\forall n\in\mathbb{N}$. Also $H(X_{n+1}\mid X_1) = H(X_n)$ shows that $(H(X_n))_{n\in\mathbb{N}}$ is (weakly) increasing. By Fekete's lemma the Avez (asymptotic) entropy exists:

$$h := \lim_{n\to\infty}\frac{H(X_n)}{n}$$

(*) In general, $H(X,Y) = H(Y) + H(Y\mid X)$, $0\leq H(Y\mid X)\leq H(Y)$.

Proposition: If the MC is symmetric, i.e., $p(x,y)=p(y,x)$ $\forall x,y\in V$, and any initial state o, one has

$$\lim_{n\to\infty} \frac{H(X_n)}{n} \geq 2\ln\frac{1}{\rho}$$

where ρ is the spectral radius. In particular for transitive symmetric chains with $\rho<1$, one has $h>0$.

Proof: By Jensen and symmetry

$$-H(X_n) = \sum_x P^n(o,x)\ln P^n(o,x) \leq \ln \sum_x [P^n(o,x)]^2 = \ln \sum_x P^n(o,x)P^n(x,o)$$
$$= \ln P^{2n}(o,o).$$

Devide by n and take liminf as $n\to\infty$. □

Theorem: For a transitive M.C. with $X_0=o$ and $H(X_1)<\infty$, the tail σ-field is P_o-trivial iff $h=0$.

Definition: Suppose P is a transition matrix on V, where $G=(V,E)$ is a graph, with corresponding graph metric d. If for all $z,w\in V$ there exists $\gamma\in Aut(V,P)\cap Aut(G)$ s.t. $\gamma(z)=w$, then the chain, with transition matrix P is called a transitive Markov chain with an invariant graph metric.

Examples: Simple random walk on a transitive graph.
Convolution random walk on a discrete group Γ, where the metric d arises from a right Cayley graph G on Γ. In this case, transitions need not be restricted to edges on G, yet the action of the group on itself by left multiplication defines automorphisms of (V,P) and G.

Consider a transitive chain on V with an invariant graph metric d determined by the transitive graph $G=(V,E)$. Fix a (base) point $o\in V$. By transitivity, the balls in G satisfy $|B(o,r+s)|\leq |B(o,r)|\cdot|B(o,s)|$, therefore Fekete's lemma implies that the volume growth exponent $v:=\lim_{r\to\infty}\frac{1}{r}\ln B(o,r)$ exists.

Similarly, if $(X_n)_{n\geq 0}$ is a transitive M.C. on V started at o, transitivity implies that $E|X_{n+m}|\leq E|X_n|+E|X_m|$, where $|x|=d(o,x)$, $x\in V$. If $E|X_1|<\infty$, Fekete's lemma again implies that the speed $l=\lim_{n\to\infty}\frac{1}{n}E|X_n|$ exists.

Theorem: Let $(X_n)_{n\geq 0}$ be a transitive M.C. on V with an invariant graph metric d, determined by the transitive graph $G=(V,E)$. Fix an initial state o.
If $E|X_1|<\infty$, then entropy, speed and volume growth satisfy $h\leq lv$.

Theorem: Let (X_n) be a transitive M.C. with $X_0=o$.
(i) If the chain is endowed with an invariant graph metric d and if $E[d(o,X_1)]<\infty$, then

$$\lim_{n\to\infty} n^{-1}d(o,X_n) = l \quad \text{a.s.}$$

(ii) If $H(X_1)<\infty$, then

$$\lim_{n\to\infty} n^{-1}\ln p_n(o,X_n) = -h.$$

Consider space $(V^{\mathbb{N}_0}, \mathcal{B})$ again.

$\mathcal{I} = \{ B \in \mathcal{B} : S^{-1}(B) = B \}$ is the invariant σ-field.

Obviously $\mathcal{I} \subseteq \mathcal{T}$, since $A \in \mathcal{I} \Rightarrow A \in \mathcal{B}$ and then $A = S^{-1}(A) \in S^{-1}(\mathcal{B})$ e.t.c., so $A \in \bigcap_{n=0}^{\infty} S^{-n}(\mathcal{B}) =: \mathcal{T}$.

For each $x \in V$, there exists a unique probability measure P_x on $(V^{\mathbb{N}_0}, \mathcal{B})$ s.t.

$$P_x(\{x_0\} \times \dots \times \{x_n\} \times V^{\{n+1, n+2, \dots\}}) = \mathbb{1}_{\{x\}}(x_0)\, p(x_0, x_1) \dots p(x_{n-1}, x_n).$$

for each elementary cylinder $\{x_0\} \times \dots \times \{x_n\} \times V^{\{n+1, n+2, \dots\}}$, $x_i \in V$, $i \in \{0, 1, \dots, n\}$, $n \in \mathbb{N}_0$.

Notation: For any measure μ on $(V^{\mathbb{N}_0}, \mathcal{B})$, $S_* \mu(B) := \mu(S^{-1}(B))$.

LEMMA: For any $x \in V$, $\sum_{y \in V} p(x,y) P_y = S_* P_x$, & $\sum_{y \in V} p^n(x,y) P_y = S_*^n P_x$ $\forall n \in \mathbb{N}_0$.

Notation: For any $x \in V$, $(V^{\mathbb{N}_0}, \mathcal{B}, P_x)$ is a probability space.
$X_0, X_1, \dots : V^{\mathbb{N}_0} \to V$ shall be the r.v. $X_i(x_0, x_1, \dots) = x_i$, $i \in \mathbb{N}_0$. (projection to i-th co-ordinate). $(X_n)_{n \in \mathbb{N}}$ shall be our stochastic process – Markov chain.

LEMMA (Markov Property): For any $x, y \in V$, $P_x(X_{n+1} = y \mid X_0, \dots, X_n) = P(X_n, y)$. P_x a.s.

Proof: On the event $\{X_0 = x, X_1 = x_1, \dots, X_n = x_n\}$ the conditional prob. equals

$$P(X_{n+1} = y \mid X_0 = x_0, \dots, X_n = x_n) = \frac{P(X_{n+1} = y, X_n = x_n, \dots, X_0 = x_0)}{P(X_n = x_n, \dots, X_0 = x_0)} = p(x_n, y).$$

DEFINITION: $u : V \to \mathbb{R}$ is P-harmonic if $u(x) = \sum_y p(x,y) u(y)$ $\forall x \in V$. $BH(V,P) = \{$bdd P-harmonic fns on $V\}$

PROPOSITION: Let P be a transition matrix on a countable state space V.
For every bdd invariant $f : V^{\mathbb{N}_0} \to \mathbb{R}$ (P_0-a.e. bounded)

$$U_f(x) := E_x[f(X_0, X_1, \dots)] \qquad x \in V$$

defines a function which is in $BH(V,P)$.
If P is irreducible, $f \mapsto U_f$ is a linear isometry between $L^\infty(V^{\mathbb{N}_0}, \mathcal{I}, P_0)$ and $BH(V,P)$. In this case the inverse of $f \mapsto U_f$ is $u \mapsto F_u$ given by

$$F_u(x_0, x_1, \dots) = \overline{\lim_{n \to \infty}}\, u(x_n) \qquad (x_0, x_1, \dots) \in V^{\mathbb{N}_0}.$$

COROLLARY: Let (V,P) be an irreducible M.C. Then the invariant σ-field is P_0 trivial iff every bdd P-harmonic function on V is constant (Liouville property)

Proof: Assume first every bdd harmonic function is constant.
Let $A \in \mathcal{I}$. Then, for $f := \mathbb{1}_A \in L^\infty(V^{\mathbb{N}_0}, \mathcal{I}, P_0)$, $U_f(x) = P_x(A)$. One must have

$$\mathbb{1}_A(x_0, x_1, \dots) = F_{U_f}(x_0, x_1, \dots) = \overline{\lim_{n \to \infty}}\, U_f(x_n) = P_0(A), \quad (\text{since } U_f(x) = U_f(0) = P_0(A), \ \forall x).$$

so $P_0(A) \in \{0,1\}$.

Conversely, suppose $P_0(A) \in \{0,1\}$ $\forall A \in \mathcal{I}$. Let $u : V \to \mathbb{R}$ be in $BH(V,P)$.
Then $F_u = \overline{\lim}\, u(x_n)$ defines a function which is $\mathcal{I}$ measurable, hence constant.
Then $U_{F_u} = u$ must be constant.

Proof of PROPOSITION:

First let $f: V^{\mathbb{N}_0} \to \mathbb{R}$ be invariant and s.t. $P_0(\{\underline{x} = (x_0, x_1, \dots) \in V^{\mathbb{N}_0} : |f(x_0, x_1, \dots)| > t\}) = 0$ for some $t > 0$, and let $\|f\|_\infty$ be the infimum of such $t > 0$. Set

$$U_f(x) := E_x[f(X_0, X_1, \dots)].$$

Then:

$$\sum_{y \in V} p(x,y)\, U_f(y) = \sum_{y \in V} p(x,y) \int_{V^{\mathbb{N}_0}} f \, dP_y$$

$$= \int_{V^{\mathbb{N}_0}} f \, d \sum_{y \in V} p(x,y) P_y$$

$$= \int_{V^{\mathbb{N}_0}} f \, d S_* P_x = \int_{V^{\mathbb{N}_0}} f \circ S \, dP_x = \int_{V^{\mathbb{N}_0}} f \, dP_x = U_f(x).$$

Assume now P irreducible.

Then Furthermore, $|U_f(x)| \le \|f\|_\infty$ P_x-a.e., because $|f(x_0, x_1, \dots)| \le \|f\|_\infty$ P_x-a.e.

Indeed: $P_0(A) = S_* P_0(A) = \sum_{y \in V} p(0,x) P_x(A)$, so $P_x(A) = 0$ whenever $p(0,x) > 0$, where A is the invariant set $\{\underline{x} \in V^{\mathbb{N}_0} : |f(\underline{x})| > \|f\|_\infty\} = A$.

By irreducibility, $p^n(0,x) > 0$ $\forall x \in V$, for some $n \in \mathbb{N}$.

Same argument yields that, for $f, g: V^{\mathbb{N}_0} \to \mathbb{R}$ invariant, $P_0(f \ne g) = 0$ (so $f = g$ in $L^\infty(V^{\mathbb{N}_0}, \mathcal{I}, P_0)$) implies that $P_x(f \ne g) = 0$ $\forall x \in V$ (in irreducible case)

so $U_f = U_g$ and $f \mapsto U_f$ defines a map on $L^\infty(V^{\mathbb{N}_0}, \mathcal{I}, P_0)$.

This map is clearly linear. So we have a map $L^\infty(V^{\mathbb{N}_0}, \mathcal{I}, P_0) \longrightarrow BH(V, P)$.

Let $u \in BH(V,P)$. Define $F_u(x_0, x_1, \dots) = \overline{\lim_{n\to\infty}}\, u(x_n)$. Let $g = F_u$.

$$U_g(x) = E_x[g(X_0, X_1, \dots)] = E_x[\overline{\lim_{n\to\infty}}\, u(X_n)] = E_x[\lim_{n\to\infty} u(X_n)] =$$
$$= \lim_{n\to\infty} E_x[u(X_n)] = u(x).$$

Reason for this:

LEMMA: $(u(X_n))_{n \in \mathbb{N}_0}$ is, under each P_x, a martingale w.r.t. filtration $\mathcal{F}_n = \sigma(X_0, \dots, X_n)$.

Proof: $E_x[u(X_{n+1}) \mid X_0, \dots X_n] = \sum_{y \in V} u(y)\, P(X_{n+1} = y \mid X_0, \dots, X_n) =$

$$\overset{P_x\text{-a.e.}}{=} \sum_{y \in V} u(y)\, p(X_n, y) = u(X_n).$$

Conversely, suppose $f: V^{\mathbb{N}_0} \to \mathbb{R}$ is invariant and P_0-a.e. bdd. Then

$$E_0[f(X_0, X_1, \dots) \mid X_0, \dots, X_n] = E_0[f(X_n, X_{n+1}, \dots) \mid X_0, \dots, X_n]$$
$$= E_{X_n}[f(X_0, X_1, \dots)] = U_f(X_n).$$

Αριστερό μέλος τείνει στο $E_0[f(X_0, X_1, \dots) \mid \sigma(X_0, X_1, \dots)] = f(X_0, X_1, \dots)$. P_0-a.e.

Hence $\lim U_f(X_n) = f(X_0, X_1, \dots)$ P_0-a.e.

Άρα $F_{U_f} = f$.

Έχουμε δείξει $U_{F_u} = u$, $F_{U_f} = f$.