Operator Algebras associated to representations of groups

Unitary Representations

H Hilbert, $\mathcal{B}(H)$ bounded operators $H \to H$.

$\mathcal{U}(H) \subseteq \mathcal{B}(H)$ unitaries. V unitary: onto and $\langle Vx, Vy\rangle = \langle x, y\rangle$ for all $x, y \in H$.

Unitary representation: (π, H) of G:

$$\pi : G \to \mathcal{U}(H)$$

group homomorphism.

The circle group

• Given $n \in \mathbb{Z}$, the map

$$\pi_n : z \to z^n : \mathbb{T} \to \mathcal{B}(\mathbb{C}) \quad (\pi_n(z)w = z^n w,\ w \in \mathbb{C})$$

is a unitary rep. of the group $\mathbb{T}$ ($\sim [0, 2\pi)$ with addition mod 2π).

• Given $f \in L^1(\mathbb{T})$, obtain $\hat{f} \in C_0(\mathbb{Z})$ by integrating $\pi_n(z)$ with 'weight' $f(z)$.

$$\hat{f}(n) = \int_{[0,2\pi]} f(e^{it})\, \overline{e^{int}}\, \frac{dt}{2\pi}.$$

Abelian groups

More generally, for G locally compact abelian with dual $\hat{G}$:

$$\hat{G} := \{\chi : G \to \mathbb{T} \text{ group homom. + continuous}\}$$

Given $f \in L^1(G)$, obtain $\hat{f} \in C_0(\hat{G})$.

$$\hat{f}(\chi) = \int_G f(t)\, \overline{\chi(t)}\, dt, \quad \chi \in \hat{G}.$$

The map $f \to \hat{f}$ (: the Fourier transform) associates functions on G to functions on dual object.

Duality (abelian groups)

$$\hat{\mathbb{Z}} = \mathbb{T} : \quad \chi_\theta(n) = e^{in\theta},\ n \in \mathbb{Z}$$
$$\hat{\mathbb{T}} = \mathbb{Z} : \quad \chi_n(e^{it}) = e^{int},\ e^{it} \in \mathbb{T}$$
$$\hat{\mathbb{R}} = \mathbb{R} : \quad \chi_y(x) = e^{2\pi ixy},\ z \in \mathbb{R}$$

Genarally: G discrete $\Rightarrow$ $\hat{G}$ compact

G compact $\Rightarrow$ $\hat{G}$ discrete

Pontryagin duality (G abelian): From the dual object $\hat{G}$ can reconstruct G: Given $t \in G$, let $\delta_t : \hat{G} \to \mathbb{T} :\ \delta_t(\chi) = \chi(t)$.

The map $G \to \hat{\hat{G}} : t \to \delta_t$ is a topological group isomorphism.

Tannaka-Krein duality does something similar for non-abelian but compact G.

From G to $L^1(G)$

$(L^1(G), *)$:

$$(f * g)(s) = \int_G f(t) g(t^{-1}s) dt$$
$$f^*(s) = \overline{f(s^{-1})}$$

From each unitary rep. (π, H) of G get *-rep. $\tilde{\pi}$ of $L^1(G)$ on same H.

$$\tilde{\pi}(f)\xi = \int_G f(t)\pi(t)\xi dt \quad \xi \in H\,.$$

Hence get C*-algebra $\overline{\tilde{\pi}(L^1(G))}$ and the von Neumann algebra by closing in a weaker topology.

Algebras

There is also a "universal" $C^*(G)$ that encodes all reps.

For compact (non-abelian) groups can get a description of $C^*(G)$ as a c_0 direct sum of (equiv. classes of) irreps.

The 'Fourier transform' for compact G

Also can define the Fourier transform as a 'matrix-valued' integral:

$$\hat{f}(\pi) = \int_G f(t)\pi(t)^* dt, \quad [\pi] \in \hat{G}.$$