Notes on the irrational rotation nonselfadjoint algebras

Aristides Katavolos
(partly based on joint work with M. Anoussis and I.G. Todorov)

As before, [1] fix $\theta \in \mathbb{R}$ s.t. $\frac{\theta}{2\pi}$ is irrational. Let $\mathcal{A} = C(\mathbb{T})$, $G = \mathbb{Z}$ and

$$(\alpha_n f)(z) = f(e^{in\theta} z) \quad (f \in \mathcal{A}, n \in \mathbb{Z}, z \in \mathbb{T}).$$

The semicrossed product $C(\mathbb{T}) \rtimes_\alpha \mathbb{Z}_+$ is a closed subalgebra of the irrational rotation algebra $C(\mathbb{T}) \rtimes_\alpha \mathbb{Z}$ (why?). [2]

Thus the representation $\pi \times \lambda : C(\mathbb{T}) \rtimes_\alpha \mathbb{Z} \to \mathcal{B}(L^2(\mathbb{T})$ restricts to an isometric representation of $C(\mathbb{T}) \rtimes_\alpha \mathbb{Z}_+$ given by (flip) [3]

$$(\pi \times \lambda)(\sum_{k=0}^{n} \delta_k \otimes f_k) = \sum_{k=0}^{n} V^k \pi(f_k)$$

where V is the generator λ_1 of $\{\lambda_n : n \in \mathbb{Z}_+\}$ given by

$$(Vg)(z) = g(e^{i\theta} z), \quad g \in L^2(\mathbb{T}).$$

The C*-algebra $C(\mathbb{T})$ is the closed algebra generated by ζ and $\bar{\zeta}$, where $\zeta(z) = z$; hence $\pi(C(\mathbb{T})) \subseteq \mathcal{B}(L^2(\mathbb{T}))$ is generated by $U := \pi(\zeta)$ and $U^* = \pi(\bar{\zeta})$. Therefore $C(\mathbb{T}) \rtimes_\alpha \mathbb{Z}_+$ is generated by $\{U, U^*, V\}$ and the crossed product $C(\mathbb{T}) \rtimes_\alpha \mathbb{Z} := \mathcal{A}_\theta$ is generated by $\{U, U^*, V, V^*\}$. They satisfy:

$$UV = e^{i\theta} VU \quad \text{(the Weyl relation)}.$$

For $\mu \in \mathbb{T}$, let $V_\mu : L^2(\mathbb{T}) \to L^2(\mathbb{T})$ be given by $V_\mu f = f_\mu$ where $f_\mu(z) = f(\mu z)$. The map $\mu \to V_\mu$ is a SOT-continuous group homomorphism into

[1] semiuv, March 29, 2015, compiled April 6, 2015

[2] This is non-trivial; one shows (see e.g. [2]) that the norm of the semicrossed product (defined as a sup over all covariant pairs (π, T) with T a contraction coincides with the (a priori smaller) norm of the crossed product, where the sup is taken over (π, U) with U unitary.

[3] This is a minor technicality.

the unitary group of $\mathcal{B}(L^2(\mathbb{T}))$; hence it is weak* continuous (because it takes values in a ball). Observe that $V_{e^{i\theta}} = V$. Note that $(\widehat{V_\mu f})(n) = \mu^n \widehat{f}(n)$. [4] More generally, if $a = (a_n)_{n\in\mathbb{Z}} \in l^\infty(\mathbb{Z})$, let D_a be given by $(\widehat{D_a f})(n) = a_n \widehat{f}(n)$; thus D_a is the image, under conjugation by the Fourier transform, of the diagonal operator on $\ell^2(\mathbb{Z})$ given by $(x_j) \to (a_j x_j)$. Let $\mathcal{D} = \{D_a : a \in \ell^\infty(\mathbb{Z})\}$. This is a masa on $L^2(\mathbb{T})$ (being unit. equivalent to the diagonal masa on $\ell^2(\mathbb{Z})$.)

l_rotgen **Lemma 1.** *The weak* closed operator algebra on $L^2(\mathbb{T})$ generated by V coincides with $\mathcal{D}$. In particular, it contains V^*.*

Proof. Since $\mu \to V_\mu$ is weak* continuous and $\{e^{in\theta} : n \in \mathbb{N}\}$ is dense in $\mathbb{T}$, the weak* closed algebra generated by the set $\{V^n : n \in \mathbb{N}\}$ equals $\{V_\mu : \mu \in \mathbb{T}\}$ and hence is selfadjoint. Since it is weak* closed, it is equal to its bicommutant which is clearly $\mathcal{D}$. □

Remark 2. *By contrast, the weak* closed operator algebra on $L^2(\mathbb{T})$ generated by U is non-selfadjoint; it is equal to $\{M_f : f \in H^\infty(\mathbb{T})\}$. In fact, the weak* closed operator algebra generated by U* and *V does not contain U^*.* [5]

Proof. Recall that $H^\infty(\mathbb{T}) = \{f \in L^\infty(\mathbb{T}) : \hat{f}(k) = 0 \ \text{for} \ k < 0\}$.

It is well-known that the Cesaro means of the Fourier series of any $f \in L^\infty(\mathbb{T})$ converges to f in the weak-* topology on $L^\infty(\mathbb{T})$ induced by $L^1(\mathbb{T})$. Thus any $f \in H^\infty(\mathbb{T})$ is a weak-* limit of polynomials in ζ (analyic polynomials) and hence M_f is a weak-* limit of polynomials in U.

On the other hand, the weak-* continuous linear form $T \to \langle T\zeta^0, \zeta^{-1}\rangle$ annihilates all polynomials in U, V (since $\langle U^k V^l \zeta^0, \zeta^{-1}\rangle = \langle U^k \zeta^0, \zeta^{-1}\rangle = 0$ when $k \geqslant 0$) but does not annihilate U^* (since $\langle U^* \zeta^0, \zeta^{-1}\rangle = 1$). □

p_nest **Proposition 3.** *The w^*-closed subalgebra of $\mathcal{B}(L^2(\mathbb{T}))$ generated by $\{U, V\}$ is the nest algebra* Alg$\mathcal{N}$ *of all operators $T \in \mathcal{B}(L^2(\mathbb{T}))$ leaving all elements of $\mathcal{N} = \{N_n : n \in \mathbb{Z}\}$ invariant, where $N_n = \{f \in L^2(\mathbb{T}) : \hat{f}(k) = 0, k < n\}$.*

Proof. Clearly, U and V belong to Alg$\mathcal{N}$, hence so does the weak* closed operator algebra that they generate.

By Lemma 1, the weak* closed algebra generated by V is equal to $\mathcal{D}$. On the other hand, if $a \in c_{00}(\mathbb{Z})$, the matrix of $U^l D_a$ with respect to the basis $\{\zeta^k\}_{k\in\mathbb{Z}}$ has the sequence a at the l-th diagonal below the main diagonal

[4] $f \to \hat{f} : L^2(\mathbb{T}) \to \ell^2(\mathbb{Z})$ is the Fourier transform.

[5] This also follows from Proposition 3, since U^* does not leave N_m invariant.

and zeros elsewhere. It follows that all lower triangular matrix units belong to the weak* closed algebra generated by U and V, and hence it equals $\mathrm{Alg}\,\mathcal{N}$. $\square$

Remark 4. *Observe that* $\mathrm{Lat}\{U,V\} = \{\zeta^k H^2 : k \in \mathbb{Z}\}$*. Thus, every invariant subspace of* $\{U,V\}$ *is actually* reduced *by the semigroup generated by* V*; hence it is invariant under the "larger" semigroup generated by* $\{U,V,V^{-1}\}$*.*

After Fourier transform $L^2(\mathbb{T}) \to \ell^2(\mathbb{Z})$:

$$U \sim \begin{bmatrix} \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \\ \dots & 0 & 0 & 0 & 0 & 0 & \dots \\ \dots & 1 & 0 & 0 & 0 & 0 & \dots \\ \dots & 0 & 1 & \mathbf{0} & 0 & 0 & \dots \\ \dots & 0 & 0 & 1 & 0 & 0 & \dots \\ \dots & 0 & 0 & 0 & 1 & 0 & \dots \\ & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}, \; V \sim \begin{bmatrix} \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \\ \dots & \bar{\lambda}^2 & 0 & 0 & 0 & 0 & \dots \\ \dots & 0 & \bar{\lambda} & 0 & 0 & 0 & \dots \\ \dots & 0 & 0 & \mathbf{1} & 0 & 0 & \dots \\ \dots & 0 & 0 & 0 & \lambda & 0 & \dots \\ \dots & 0 & 0 & 0 & 0 & \lambda^2 & \dots \\ & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

We write $\mathcal{A}_\theta^+$ and $\mathcal{A}_\theta^{++}$ for the norm-closed subalgebras of $\mathcal{A}_\theta$ generated by $\{U,V,V^*\}$ and $\{U,V,I\}$ respectively.

Now $\mathcal{N}$ becomes the $\mathbb{Z}$-ordered nest on $\ell^2(\mathbb{Z})$ with non-trivial elements N_m, $m \in \mathbb{Z}$ where $N_m = [e_k : k \geqslant m]$; thus $U(N_m) = N_{m+1} \subset N_m$ and $V(N_m) = N_m$. It follows that U, V and V^* lie in the nest algebra $Alg(\mathcal{N})$ and so

$$\mathcal{A}_\theta^{++} \subset \mathcal{A}_\theta^+ \subseteq \mathcal{A}_\theta \cap Alg(\mathcal{N}).$$

We have shown in Proposition 3 that the weak-* closure of $\mathcal{A}_\theta^{++}$ is the whole of $Alg(\mathcal{N})$, and so the same is true for the w*-closure of $\mathcal{A}_\theta^+$ (but this might be obvious anyway). Thus

$$W^*(\mathcal{A}_\theta^{++}) = W^*(\mathcal{A}_\theta^+) = Alg(\mathcal{N}).$$

On the other hand, since $\mathcal{A}_\theta$ is an irreducible C*-algebra, its w* closure is $B(H)$.

For the proof of the following Proposition, we shall need a conditional expectation $\Psi : \mathcal{A}_\theta \to C^*(V)$. This is constructed as follows (see [1, Theorem VI.1.1] for more details):

There is a *-automorphism ρ_t of $\mathcal{A}_\theta$ given on the generators by $\rho_t(U) = e^{it}U$ and $\rho_t(V) = V$. Moreover for all $a \in \mathcal{A}_\theta$ the map $t \to \rho_t(a)$ is (norm-) continuous. Thus the integral

$$\Psi(a) = \frac{1}{2\pi}\int_0^{2\pi} \rho_t(a)dt$$

exists. It is easy to see that Ψ is linear positive unital and $\|\Psi\| = 1$.

On easily verifies that if $a = \sum_{k,l} c_{k,l}V^kU^l$ is a finite sum,

$$\Psi(a) = \sum_{k,0} c_{k,0}V^kU^0.$$

It follows by continuity of Ψ that it is an idempotent mapping $\mathcal{A}_\theta$ onto the C*-subalgebra $C^*(V)$ generated by V; it is *a conditional expectation.*

The following formula holds for all $a \in \mathcal{A}_\theta$

$$\Psi(a) = \lim_n \frac{1}{2n+1}\sum_{|k|\leqslant n} V^kaV^{-k}.$$

This can be verified when a is a finite sum as above; [6] hence it is valid on the whole of $\mathcal{A}_\theta$ by continuity.

Any $a \in \mathcal{A}_\theta$ has a formal expansion:

$$a \sim \sum_{n\in\mathbb{Z}} \Psi(aU^{-n})U^n.$$

When $a = \sum_{k,l} c_{k,l}V^kU^l$ is a finite sum, one verifies that the above formula is in fact an equality.

For general $a \in \mathcal{A}_\theta$, the means of the partial sums sum $s_k(a) = \sum\limits_{|n|\leqslant k} \Psi(aU^{-n})U^n$ converge in norm to a. Indeed,

$$\begin{aligned}
\sigma_m(a) &:= \frac{1}{m+1}(s_0(a) + \cdots + s_m(a)) \\
&= \sum_{|n|\leqslant m}\left(1 - \frac{|n|}{m+1}\right)\Psi(aU^{-n})U^n \\
&= \sum_{|n|\leqslant m}\left(1 - \frac{|n|}{m+1}\right)\int_0^{2\pi} \rho_t(aU^{-n})\frac{dt}{2\pi}\,U^n \\
&= \sum_{|n|\leqslant m}\left(1 - \frac{|n|}{m+1}\right)\int_0^{2\pi} \rho_t(a)e^{-int}\frac{dt}{2\pi} = \int_0^{2\pi} \rho_t(a)k_n(t)\frac{dt}{2\pi}
\end{aligned}$$

[6] use the fact that $\lim\limits_n \frac{1}{2n+1}\sum_{|k|\leqslant n} e^{iks} = 0$ unless $s = 0$

where k_n is Féjer's kernel. We all know that if f is continuous (even Banach-space valued), then $\int_0^{2\pi} f(t)K_n(t)\frac{dt}{2\pi} \to f(0)$ and so $\sigma_n(a) \to \rho_0(a) = a$.

Proposition 5. *We have $\mathcal{A}_\theta^+ = \mathcal{A}_\theta \cap Alg(\mathcal{N})$. In other words $\mathcal{A}_\theta^+$ is a nest subalgebra of a C^*-algebra.*

Proof. Suppose that $a \in \mathcal{A}_\theta \cap Alg(\mathcal{N})$, so $a(N_m) \subseteq N_m$ for all $m \in \mathbb{Z}$.

In order to show that $a \in \mathcal{A}_\theta^+$, it suffices by the discussion above to show that in the expansion $\sum_{n\in\mathbb{Z}} \psi(aU^{-n})U^n$ the terms $\psi(aU^{-n})U^n$ vanish when $n < 0$.

Claim. Since $a(N_m) \subseteq N_m$, the same holds for each monomial $\psi(aU^{-n})U^n$.

Proof. Recall that $\psi(aU^{-n})$ is the limit of convex sums of terms $V^k aU^{-n}V^{-k}$ for which we have

$$\begin{aligned} V^k aU^{-n}V^{-k}(N_m) &= V^k aU^{-n}(N_m) \subseteq V^k a(N_{m-n}) \\ &\subseteq V^k(N_{m-n}) = N_{m-n} \end{aligned}$$

and therefore $\psi(aU^{-n})(N_m) \subseteq N_{m-n}$ so that
$\psi(aU^{-n})U^n(N_m) \subseteq \psi(aU^{-n})N_{m+n} \subseteq N_m$ as claimed.

Now $\psi(aU^{-n}) \in C^*(V)$; write $\psi(aU^{-n}) = f_n(V)$ for some continuous function f_n on the spectrum of V (which is actually the whole of $\mathbb{T}$; why?).

By the Claim, for each $m \in \mathbb{Z}$, we have $f_n(V)U^n e_m \in N_m$ and so $\langle f_n(V)U^n e_m, e_{m+p}\rangle = 0$ when $p < 0$. But $f_n(V)U^n e_m = f_n(V)e_{n+m} = f_n(\beta^{n+m})e_{n+m}$ (where we write $\beta = e^{i\theta}$) and so we obtain

$$0 = \langle f_n(V)U^n e_m, e_{m+p}\rangle = \left\langle f_n(\beta^{m+n})e_{m+n}, e_{m+p}\right\rangle = f_p(\beta^{m+p}).$$

Since this holds for all $m \in \mathbb{Z}$ and θ is irrational and f_p is continuous, it follows that f_p must vanish identically. Thus $f_p(V) = \psi(aU^{-p}) = 0$ for all $p < 0$, and so $a \sim \sum_{n\geqslant 0} \psi(aU^{-n})U^n$ is in $\mathcal{A}_\theta^+$. ☐

Proposition 6. *The inclusion $\mathcal{A}_\theta^{++} \subset \mathcal{A}_\theta^+$ is proper.*

Proof. If $p(U,V) = \sum_{k,l\geqslant 0} c_{k,l}U^k V^l$ is a polynomial in $\mathcal{A}_\theta^{++}$, then the diagonal of $V^* - p(U,V)$ is the operator $V^* - \sum_{l\geqslant 0} c_{0,l}V^n$ whose norm is at least 1. Thus $\|V^* - p(U,V)\| \geqslant 1$ for every polynomial in $\mathcal{A}_\theta^{++}$, showing that V^* is (in $\mathcal{A}_\theta^+$ but) not in $\mathcal{A}_\theta^{++}$.

In more detail, for each $n \in \mathbb{Z}$ (writing $\beta = e^{i\theta}$ again)

$$\begin{aligned}
\langle (V^* - p(U,V))e_n, e_n \rangle &= \langle V^* e_n, e_n \rangle - \sum_{k,l \geqslant 0} c_{k,l} \left\langle U^k V^l e_n, e_n \right\rangle \\
&= \left\langle \bar{\beta}^n e_n, e_n \right\rangle - \sum_{k,l \geqslant 0} c_{k,l} \left\langle U^k \beta^{ln} e_n, e_n \right\rangle \\
&= \bar{\beta}^n - \sum_{k,l \geqslant 0} c_{k,l} \beta^{ln} \langle e_{n+k}, e_n \rangle = \bar{\beta}^n - \sum_{l \geqslant 0} c_{0,l} \beta^{ln}
\end{aligned}$$

and thus

$$\|V^* - p(U,V)\| \geqslant \sup_n |\langle (V^* - p(U,V))e_n, e_n \rangle| = \sup_n |\bar{\beta}^n - \sum_{l \geqslant 0} c_{0,l} \beta^{ln}|.$$

But since $\{\beta^n : n \in \mathbb{Z}\}$ is dense in $\mathbb{T}$, the last supremum is the same as $\sup\{|\bar{z} - \sum_{l \geqslant 0} c_{0,l} z^l| : z \in \mathbb{T}\}$. But this is at least 1. [7] Therefore finally

$$\|V^* - p(U,V)\| \geqslant 1$$

for any polynomial $p(U,V)$, and so V^* is not in the closure $\mathcal{A}_\theta^{++}$ of such polynomials. □

References

[1] Kenneth R. Davidson. *C^*-algebras by example*, volume 6 of *Fields Institute Monographs*. American Mathematical Society, Providence, RI, 1996.

[2] Justin Peters. Semicrossed products of C^*-algebras. *J. Funct. Anal.* 59(3):498–534, 1984.

[7] The functions $\zeta_k(z) = z^k$ are orthonormal in $L^2(\mathbb{T})$, and so

$$\begin{aligned}
\sup\{|\bar{z} - \sum_{l \geqslant 0} c_{0,l} z^l| : z \in \mathbb{T}\}^2 &= \left\| \zeta_{-1} - \sum_{l \geqslant 0} c_{0,l} \zeta_l \right\|_\infty^2 \\
&\geqslant \left\| \zeta_{-1} - \sum_{l \geqslant 0} c_{0,l} \zeta_l \right\|_2^2 = \|\zeta_{-1}\|_2^2 + \sum_{l \geqslant 0} c_{0,l} \|\zeta_l\|_2^2 \geqslant 1.
\end{aligned}$$