Ergodicity and Operator Algebras: the Murray - von Neumann examples

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Aristides Katavolos

von Neumann algebras

If $\mathscr{S} \subseteq \mathscr{B}(H)$, its **commutant** $\mathscr{S}'$ consists of all $T \in \mathscr{B}(H)$ satisfying $TS = ST$ for all $S \in \mathscr{S}$. It is clear that $\mathscr{S}'$ is always a unital algebra, closed in the weak operator topology (WOT): that is, if $T_i \in \mathscr{S}'$ and $\langle T_i x, y\rangle \to \langle Tx, y\rangle$ for all $x, y \in H$, then $T \in \mathscr{S}$. Also, if $\mathscr{S}$ is selfadjoint, so is $\mathscr{S}'$.

A **von Neumann algebra** is a selfadjoint unital subalgebra of $\mathscr{B}(H)$ which is closed in the WOT topology.

Theorem (von Neumann's bicommutant theorem)

If $\mathscr{A} \subseteq \mathscr{B}(H)$ is a selfadjoint unital algebra and $T \in \mathscr{B}(H)$, the following are equivalent:
(a) $T \in \mathscr{A}''$.
(b) For each $\xi \in \mathscr{H}$, the operator T is in the closed linear span of $\{Ax : A \in \mathscr{A}\}$.
(c) T is in the WOT-closure of $\mathscr{A}$.

The von Neumann algebra of a group

Let [1] G be a countable (discrete) group. (Think of $\mathbb{Z}$ or $\mathbb{F}_2$.)

$$H = \ell^2(G) = \{f : G \to \mathbb{C} : \sum_{t \in G} |f(t)|^2 < \infty\}.$$

Then $\ell^2(G)$ has ON basis $\{\delta_t : t \in G\}$.
For $s \in G$ define a map

$$\lambda_s : \delta_t \to \delta_{st}$$

and extend linearly. This is an ℓ^2 isometry, so extends to $\lambda_s : \ell^2(G) \to \ell^2(G)$. But it is onto because $\lambda_s \lambda_t = \lambda_{st}$ so $\lambda_s \lambda_{s^{-1}} = I$, hence unitary.

$$\text{For } f \in \ell^2(G), \quad (\lambda_s f)(t) = f(s^{-1} t).$$

[1] fact15p, January 21, 2015

The von Neumann algebra of a group

Definition

The von Neumann algebra generated by the set of unitaries

$$\{\lambda_t : t \in G\}$$

is called the von Neumann algebra $\mathrm{vN}(G) = \mathscr{L}(G)$ *of the group.*

Note that

$$\mathrm{vN}(G) = \overline{\mathrm{span}\{\lambda_t : t \in G\}}^{wot}$$

by the bicommutant theorem (because this span is a unital *-algebra).

The von Neumann algebra of a group

If $A \in B(H)$ is a *finite sum*

$$A = \sum_{u \in G} f_A(u) \lambda_u$$

then its matrix

$$a_{s,t} = \langle A\delta_t, \delta_s \rangle = \sum_{u \in G} f_A(u) \langle \delta_{ut}, \delta_s \rangle = f_A(st^{-1})$$

is 'constant along diagonals'. This formula remains true for any $A \in \mathrm{vN}(G)$; now $f_A \in \ell^2(G)$ (because $A\delta_e = \sum_u f_A(u)\delta_u \in \ell^2(G)$). The map $A \to f_A$ is linear, 1-1, contractive, dense range, but not onto $\ell^2(G)$. (Consider e.g. $\mathbb{Z}$.)

The commutant

The **commutant** $(\mathscr{L}(G))'$ of $\mathscr{L}(G)$

$$(\mathscr{L}(G))' := \{T \in B(H) : TA = AT \ \forall A \in \mathscr{L}(G)\}$$

is $\mathscr{R}(G)$, the von Neumann algebra generated by all *right translations* ρ_t, $t \in G$ where $(\rho_t f)(s) = f(st)$, $f \in H$. is this OK?

Proof. Note that

$$\begin{aligned} \rho_s(\lambda_t \delta_u) &= \rho_s(\delta_{tu}) = \delta_{(tu)s} \\ \lambda_t(\rho_s \delta_u) &= \lambda_t(\delta_{us}) = \delta_{t(us)} \\ \text{so } \ \rho_s \lambda_t &= \lambda_t \rho_s. \end{aligned}$$

Thus $\lambda_t \in (\mathscr{R}(G))'$ and so $\mathscr{L}(G) \subseteq (\mathscr{R}(G))'$. Conversely, $\rightsquigarrow$

The commutant

take $T \in (\mathscr{R}(G))'$. To show $T \in \mathscr{L}(G)$.
Let $x = T\delta_e \in \ell^2(G)$.
Given $y \in H$, we define

$$x * y = \sum_{h,g} x(g)y(h)\delta_{gh}$$

as a formal series. Then

$$(x*y)(k) = \sum_{h,g} x(g)y(h)\delta_{gh}(k) = \sum_{g} x(g)y(g^{-1}k) = \sum_{h} x(kh^{-1})y(h).$$

Since clearly $\sum_g |y(g^{-1}k)|^2 = \sum_h |y(h)|^2 = \|y\|_2^2$ and similarly $\sum_h |x(kh^{-1})|^2 = \|x\|_2^2$, it follows that for all $k \in G$,

$$|(x*y)(k)| \leq \|x\|_2 \|y\|_2 \qquad \text{for all } k \in G. \tag{1}$$

The commutant

Note that for all $g, h \in G$,

$$(x * \delta_g)(h) = \left\langle x * \delta_g, \delta_h \right\rangle = \left\langle \rho_g(x), \delta_h \right\rangle = \left\langle \rho_g(T\delta_e), \delta_h \right\rangle$$
$$= \left\langle T(\rho_g \delta_e), \delta_h \right\rangle = \left\langle T\delta_g, \delta_h \right\rangle$$

Thus, for fixed $h \in G$, the mappings $y \to (x * y)(h)$ and $y \to \langle Ty, \delta_h \rangle$ agree when y runs through the orthonormal basis $\{\delta_g : g \in G\}$. Since they are both continuous and linear, they must agree on H, and so for all $y \in H$ we have

$$(x * y)(h) = \langle Ty, \delta_h \rangle \qquad \text{or} \qquad (x * y)(h) = (Ty)(h)$$

for all $h \in G$, that is $Ty = x * y$ for all $y \in H$. It follows that $T \in \mathscr{L}(G)$. Indeed, for all $X \in \mathscr{L}(G)'$ and all $t \in G$,

$$XT\delta_t = X(x * \delta_t) = X\left(\sum_s x(s)\delta_{st}\right) = \sum_s x(s)X\delta_{st}$$
$$= \sum_s x(s)X\lambda_s\delta_t = \sum_s x(s)\lambda_s(X\delta_t) = x * (X\delta_t) = TX\delta_t.$$

The trace

The **trace** is the linear functional τ defined on $\mathrm{vN}(G)$ by

$$\tau(A) = \langle A\delta_e, \delta_e\rangle \quad \text{for all } A \in \mathrm{vN}(G).$$

It is a WOT-continuous state, it is *faithful* because
$\tau(A^*A) = 0 \iff A\delta_e = 0 \iff A\delta_t = A\rho_t\delta_e = 0 \iff A = 0$
(δ_e *separates* $\mathrm{vN}(G)$) and it is a *trace*, i.e.

$$\tau(AB) = \tau(BA) \qquad \text{for all } A, B \in \mathrm{vN}(G).$$

(enough to check this when $A = \lambda_s, B = \lambda_t$: obvious!).

Example of a non-type I factor: $\mathrm{vN}(F_2)$

The *centre* $\mathscr{L}(G)\cap(\mathscr{L}(G))'$ consists of all (if any!) $A\in\mathscr{L}(G)$ such that f_A is constant on conjugacy classes $C_t=\{sts^{-1}: s\in G\}$.

Example But when $G=F_2$, all $C_t\,(t\neq e)$ are infinite; and since $f_A\in\ell^2(F_2)$, it must be constant!

Conclusion: $\mathrm{vN}(F_2)$ is a *factor*, like $B(H)$.
But it has a finite faithful trace, with $\tau(\mathrm{vN}(G)_+)=[0,1]$, unlike $B(\ell^2)$.

For example, if $u\in\mathrm{vN}(G)$ is isometric, it is unitary.

The group-measure space construction (Murray - von Neumann)

Let $(X, \mathscr{S}, \mu)$ be a *countably separated* measure space.
Let G be a countable group acting on X by measure-class preserving bijections $\{\phi_t : t \in G\}$.
So can define

$$U_t : L^2(\mu) \to L^2(\mu) : \; U_t f = r_t(f \circ \phi_t)$$

where $r_t = \sqrt{\dfrac{d\mu \circ \phi_t}{d\mu}} > 0$ μ-a.e. (makes U_t isometric).
Represent $L^\infty(\mu)$ and G on $H = L^2(\mu) \otimes L^2(G) \simeq L^2(X \times G)$ by

$$\pi(f) = M_f \otimes I \quad \text{and} \quad W_t = U_t \otimes \lambda_t$$

Define the crossed product $\mathscr{A} = L^\infty(\mu) \rtimes G$ by t

$$\mathscr{A} = \{\pi(f), W_t : f \in L^\infty(\mu), t \in G\}'' = \overline{\{\sum_k \pi(f_k) W_{t_k} : f_k \in L^\infty, t_k \in G\}}^{w*}$$

Examples of factors

Assumptions on action:

(1) The action of G on (X,μ) is called an (essentially) **free action** if for each $t \in G$, $t \neq e$, the fixed point set $F_t := \{x \in X : \phi_t(x) = x\}$ has $\mu(F_t) = 0$.

(2) The action of G on (X,μ) is called **ergodic** if

$$f \in L^\infty(\mu), f \circ \phi_t = f \ \forall t \in G \quad \Rightarrow \quad f = cst.$$

(Equiv: (almost) invariant measurable sets are null or conull)

Proposition

Under these two assumptions $\mathscr{A}$ is a factor (trivial centre).

Examples of factors

(I) Let $X = \mathbb{Z}$ with counting measure, $G = \mathbb{Z}$ and $\phi_n(k) = k + n$ (: transitive action); then:

$$\mathscr{A} \simeq \mathscr{B}(\ell^2(\mathbb{Z})).$$

(I_n) (Variation of (I)): Now $X = G = \mathbb{Z}_n$ (finite cyclic group); obtain

$$\mathscr{A} \simeq \mathscr{B}(\ell^2(\mathbb{Z}_n)) \simeq M_n.$$

(II_1) Let $(X, \mu) = (\mathbb{T}, m)$, let $G = \mathbb{Z}$ and $\phi_n(z) = e^{2\pi i n\theta} z$ where $\theta \notin \mathbb{Q}$ (:G preserves a finite measure).
$\mathscr{A}$ has a normal faithful finite trace τ with $\tau(\mathscr{P}(\mathscr{A})) = [0, 1]$.

(II_∞) Let $(X, \mu) = (\mathbb{R}, m)$, let $G = \mathbb{Q}$ and $\phi_q(x) = x + q$ (: G preserves only an infinite but σ-finite measure).
$\mathscr{A}$ has a normal faithful semifinite trace τ with $\tau(\mathscr{P}(\mathscr{A})) = [0, \infty]$.

(III) $(X, \mu) = (\mathbb{R}, m)$, let $G = \{\phi_{n,q} : n \in \mathbb{Z}, q \in \mathbb{Q}\}$ where, for a fixed $a > 1$, $\phi_{n,q}(x) = a^n x + q$ (: G preserves no σ-finite measure).