Crossed products of Operator Algebras

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Reminder

$$A: \quad \text{C*-algebra} \qquad G: \quad \text{discrete group}$$
$$t \to \alpha_t : G \to Aut(A) \quad \text{group homomorphism}$$
$$\alpha_t(a) = t \cdot a$$

A (discrete) **C*-dynamical system** is a triple (A, α, G) where $\alpha : G \to Aut(A)$ is a group morphism into the group of *-automorphisms of A.

Definition

A **covariant representation** of a C*-dynamical system $(\mathscr{A}, \alpha, G)$ on a Hilbert space H is a pair $(\pi, U : H)$ where π is a *-representation of $\mathscr{A}$ on H, U is a unitary representation of G on the same H and π and U are connected by the *covariance condition*:

$$\pi(\alpha_g(a)) = U_g \pi(a) U_g^* \qquad (a \in \mathscr{A}, g \in G). \qquad \text{(C)}$$

Example

Let Ω be a locally compact Hausdorff space, G a group of homeomorphisms of Ω, μ a G-invariant Borel measure on Ω (thus $\mu(tS) = \mu(S)$ for all $t \in G$ and $S \subseteq \Omega$ Borel).
Let $A = C_0(\Omega)$ and $\alpha_t(a) = a \circ t^{-1}$.
Represent A on $H = L^2(\Omega, \mu)$ as multiplication operators:

$$\pi(a)f = af \qquad (a \in A,\ f \in H).$$

Represent G on H *by composition:*

$$U_t f = f \circ t^{-1}$$

(the fact that each U_t is unitary follows from the fact that μ is G-invariant).
The pair (π, U) is covariant.

The twisted convolution algebra

$$A \otimes c_{00}(G) = c_{00}(G; A) = \{f : G \to A : \operatorname{supp} f \;\text{ finite}\}$$

This is the linear span of the functions $a \otimes f, a \in A, f \in c_{00}(G)$ where

$$(a \otimes f)(t) = af(t) \in A$$

It is also the linear span of the functions $a \otimes \delta_s, a \in A, s \in G$ where

$$(a \otimes \delta_s)(t) = \begin{cases} a, & t = s \\ 0, & t \neq s \end{cases}$$

$$\text{So} \quad f = \sum_t f(t) \otimes \delta_t \,.$$

Given covariant pair $(\pi, U : H)$, define $(\pi \times U)(a \otimes \delta_s) = \pi(a)U_s$, i.e.

$$(\pi \times U)\left(\sum_t f(t) \otimes \delta_t\right) := \sum_t \pi(f(t))U_t \in B(H)$$

The twisted convolution algebra

Want to define *-algebra structure on $A \otimes c_{00}(G)$ making $\pi \times U$ a *-representation: covariance requires

$$\pi(a)U_s\pi(b)U_r = \pi(a)\pi(\alpha_s(b))U_sU_r, \text{ so}$$

$$\begin{aligned}(a \otimes \delta_s) * (b \otimes \delta_r) &= (a\alpha_s(b)) \otimes \delta_{sr} \\ \text{i.e.} \quad (\phi * \psi)(t) &= \sum_{sr=t} \phi(s)\alpha_s(\psi(r)) \\ &= \sum_{s \in G} \phi(s)\alpha_s(\psi(s^{-1}t)).\end{aligned}$$

and

$$(\pi(a)U_s)^* = U_{s^{-1}}\pi(a^*) = \pi(\alpha_{s^{-1}}(a^*))U_{s^{-1}}, \text{ so}$$

$$\begin{aligned}(a \otimes \delta_s)^* &= \alpha_{s^{-1}}(a^*) \otimes \delta_{s^{-1}} \\ \text{i.e.} \quad \phi^*(t) &= \alpha_t(\phi^*(t^{-1})).\end{aligned}$$

The (full) crossed product

Definition

The completion of the twisted convolution algebra $(A \otimes c_{00}(G), *)$ *with respect to*

$$\|f\| := \sup\{\|(\pi \times U)(f)\| : (\pi, U : H) \text{ covariant rep.}\} \qquad (*)$$

is called the (full) crossed product $A \rtimes_\alpha G$.

It is a C*-seminorm, but why a norm?

Existence of covariant representations

For each *-rep $\pi : A \to \mathscr{B}(H_0)$ define
$H = H_0 \otimes \ell^2(G) \cong \ell^2(G, H_o)$.
Define a representation $\tilde{\pi}$ of A on H by

$$\begin{aligned} \tilde{\pi}(a)(x \otimes \delta_s) &= \pi(\alpha_{s^{-1}} a)x \otimes \delta_s, \quad \text{i.e. } \tilde{\pi}(a) = \operatorname{diag}(\pi(\alpha_{s^{-1}} a)) \\ (\tilde{\pi}(a)\xi)(t) &= \pi(\alpha_{t^{-1}}(a))(\xi(t)) \qquad (a \in A, \xi \in \ell^2(G, H_0)). \end{aligned} \tag{1}$$

Define a unitary representation Λ of G on H by

$$\begin{aligned} \Lambda_s(x \otimes \delta_t) &= x \otimes \delta_{st}, \quad \text{i.e.} \\ (\Lambda_s \xi)(t) &= \xi(s^{-1}t) \qquad (s \in G, \xi \in \ell^2(G, H_o)). \end{aligned} \tag{2}$$

This is covariant; and if π is faithful on A, then $\tilde{\pi} \times \Lambda$ is faithful on the convolution algebra $A \otimes c_{00}(G)$.

$\tilde{\pi} \times \Lambda$ is faithful on $A \otimes c_{00}(G)$

Indeed, if $f = \sum_t f(t) \otimes \delta_t \in A \otimes c_{00}(G)$, then, for each $x \in H_0$,

$$
\begin{aligned}
(\tilde{\pi} \times \Lambda)(f)(x \otimes \delta_1) &= \left(\sum_t \tilde{\pi}(f(t))\Lambda_t \right) (x \otimes \delta_1) \\
&= \sum_t \tilde{\pi}(f(t))(x \otimes \delta_t) \\
&= \sum_t \pi(\alpha_{t^{-1}}(f(t)))x \otimes \delta_t \\
\Rightarrow \ \|(\tilde{\pi} \times \Lambda)(f)(x \otimes \delta_1)\|^2 &= \sum_t \|\pi(\alpha_{t^{-1}}(f(t)))x\|^2 \qquad (3)
\end{aligned}
$$

hence if $(\tilde{\pi} \times \Lambda)(f) = 0$ then for each $x \in H_0$ and $t \in G$ we have $\pi(\alpha_{t^{-1}}(f(t)))x = 0$ and so each $f(t)$ vanishes since $\pi \circ \alpha_{t^{-1}}$ is injective.

Existence of covariant representations

Thus (*) defines a norm $\|\cdot\|$ on $A \otimes c_{00}(G)$.
The completion of $A \otimes c_{00}(G)$ with respect to the (a priori smaller) norm

$$\|f\|_r := \|(\tilde{\pi} \times \Lambda)(f)\|$$

is called *the reduced crossed product $A \rtimes_r G$*.
It coincides with $A \rtimes G$ when G is abelian, or compact, but not necessarily when $G = \mathbb{F}_2$.

Example

If $G = \mathbb{Z}$, $A = \mathbb{C}$ and α is the trivial action, then the unitary $V := \Lambda_1$ is just the bilateral shift on $\ell^2(\mathbb{Z})$, which is unitarily equivalent to multiplication by z on $L^2(\mathbb{T})$. If π is the identity representation of $\mathbb{C}$ as operators on $\mathbb{C}$, then the representation $\tilde{\pi} \times \Lambda$ extends to a faithful representation of $A \rtimes_{id} \mathbb{Z}$ on $L^2(\mathbb{T})$.
If $\phi = \sum \phi_k \otimes \delta_k$ is in $c_{oo}(\mathbb{Z})$, then $(\tilde{\pi} \times \Lambda)(\phi) = \sum \phi_k V^k$ is the operator of multiplication by the function $\sum \phi_k z^k$, whose norm is precisely the supremum norm of the function.
Since such functions are dense in $C(\mathbb{T})$, it follows that $\mathbb{C} \times_{id} \mathbb{Z}$ is isometrically isomorphic to $C(\mathbb{T})$.
The dense subalgebra $\ell^1(\mathbb{Z})$ of $\mathbb{C} \times_{id} \mathbb{Z}$ is mapped by $\tilde{\pi} \times \Lambda$ to the Wiener algebra, that is the algebra of all $f \in C(\mathbb{T})$ whose Fourier series is absolutely convergent.

Fourier coefficients

For $\phi = \sum_t \phi_t \otimes \delta_t \in A \otimes c_{00}(G)$ call ϕ_t **the t-th Fourier coefficient** of ϕ.
Fix a faithful rep. π_0 of A. Note that by (3), $\|\pi_0(\alpha_{t^{-1}}(\phi_t))\| \leq \|(\tilde{\pi}_0 \times \Lambda)(\phi)\|$ for each $t \in G$. Now

$$\begin{aligned}\|\phi_s\|_A &= \|\pi_0(\alpha_{s^{-1}}(\phi_s))\| \\ &\leq \sup\{\|(\pi \times U)(\phi)\| : \pi \times U \text{ covariant pair}\} = \|\phi\|.\end{aligned}$$

Hence the map

$$E_s : A \otimes c_{00}(G) \to A : \phi \to \phi_s$$

is contractive, so extends to a contraction

$$E_s : A \rtimes_\alpha G \to A.$$

Clearly if $a \in A \rtimes_\alpha G$ has $(\tilde{\pi}_0 \times \Lambda)(a) = 0$ then each $E_s(a) = 0$.
Hence if the "Fourier transform" is injective, the reduced crossed product coincides with the full crossed product.

Abelian groups

For G abelian, let $\Gamma = \widehat{G} = \{\gamma : G \to \mathbb{T} : \text{cts homom.}\}$ be the dual group. For $\gamma \in \Gamma$, let

$$\theta_\gamma(\sum_g \phi_g \otimes \delta_g) = \sum_g \phi_g \otimes \gamma(g)\delta_g.$$

Each θ_γ extends to an isometric *-automorphism of $\mathscr{A} \times_\alpha G$ and

$$E_t(B) \otimes \delta_t = \int_\Gamma \theta_\gamma(B)\gamma(t^{-1})d\gamma \quad \forall B \in \mathscr{A} \times_\alpha G, \forall t \in G. \qquad (*)$$

Let ξ be a continuous linear form on $\mathscr{A} \times_\alpha G$. Let $f(\gamma) = \xi(\theta_\gamma(B))$; its Fourier transform is

$$\hat{f}(t) = \int_\Gamma f(\gamma)\gamma(t^{-1})d\gamma = \xi(E_t(B) \otimes \delta_t).$$

So if each $E_t(B)$ is zero, then $\hat{f} = 0$ and so $f = 0$; therefore $B = 0$.

Abelian groups

Note that $\theta_\gamma(a \otimes \delta_e) = a$ when $a \in \mathscr{A}$ hence $E_e(a \otimes \delta_e) = a$ by $(*)$.
Identify $\mathscr{A}$ with its image $\{a \otimes \delta_e : a \in \mathscr{A}\}$ in $\mathscr{A} \times_\alpha G$. The map

$$E_e : \mathscr{A} \times_\alpha G \to \mathscr{A}$$

is a contractive projection, and

$$\begin{aligned} E_e(aBc) &= E_e(aBc) \otimes \delta_e = \int_\Gamma \theta_\gamma(aBc)\gamma(e^{-1})d\gamma = \int_\Gamma a\theta_\gamma(B)cd\gamma \\ &= a(E_e(B))c \end{aligned}$$

conditional expectation. Also, faithful:

$$0 = E_e(B^*B) = \int_\Gamma \theta_\gamma(B^*B)d\gamma \Rightarrow B^*B = 0 \Rightarrow B = 0$$

because $\gamma \to \theta_\gamma(B^*B)$ is nonneg. and continuous.

Universal property

There exists a C*-algebra $\mathscr{B}$ satisfying

(a) There exist embeddings $i_A : \mathscr{A} \to \mathscr{B}$ (a *-representation, necessarily 1-1) and $i_G : G \to \mathscr{U}(\mathscr{B})$ (a - necessarily injective- group homomorphism into the unitary group $\mathscr{U}(\mathscr{B})$ of $\mathscr{B}$) satisfying
$i_A(\alpha_s(x)) = i_G(s) i_A(x) i_G(s)^*$ for all $x \in \mathscr{A}, s \in G$;

(b) for every covariant representation $(\pi, U; H)$ of $(\mathscr{A}, G, \alpha)$, there is a non-degenerate representation $\pi \times U$ of $\mathscr{B}$ with $\pi = (\pi \times U) \circ i_A$ and $U = (\pi \times U) \circ i_G$;

(c) the linear span of $\{i_A(x) i_G(s) : x \in \mathscr{A}, s \in G\}$ is dense in $\mathscr{B}$.

This C*-algebra $\mathscr{B}$ is unique (up to *-isomorphism) and is the crossed product $\mathscr{A} \rtimes_\alpha G$.

Example: The irrational rotation algebra

Fix $\theta \in \mathbb{R}$ s.t. $\frac{\theta}{2\pi}$ is irrational and write $\lambda = e^{i\theta}$. Let $\mathscr{A} = C(\mathbb{T})$, $G = \mathbb{Z}$ and

$$(\alpha_n f)(z) = f(\lambda^n z) \quad (f \in \mathscr{A}, n \in \mathbb{Z}, z \in \mathbb{T}).$$

The crossed product $C(\mathbb{T}) \rtimes_\alpha \mathbb{Z} := \mathscr{A}_\theta$ is called the irrational rotation algebra.
The reduced representation on $H = L^2(\mathbb{T}) \otimes \ell^2(\mathbb{Z})$:

$$(\tilde{\pi} \times \Lambda) \left(\sum_{|k| \leq n} f_k \otimes \delta_k \right) = \left(\sum_{|k| \leq n} \tilde{\pi}(f_k) \Lambda_k \right)$$

where $\pi : C(\mathbb{T}) \to B(L^2(\mathbb{T})) : \ \pi(f)g = fg \, (g \in L^2(\mathbb{T}))$

(see (1) and (2)) is faithful on $C(\mathbb{T}) \rtimes_\alpha \mathbb{Z}$ since $\mathbb{Z}$ is abelian.

The irrational rotation algebra

But the representation on $L^2(\mathbb{T})$ given by

$$(\pi \times \lambda)\left(\sum_{|k|\le n} f_k \otimes \delta_k\right) = \left(\sum_{|k|\le n} \pi(f_k)\lambda_k\right)$$

(where $\lambda_k = U^k$ with $U(\delta_k) = \delta_{k+1}$ the bilateral shift) is also faithful because Lebesgue measure is ergodic for the irrational rotation.
So we have two isometric representations of the same C* algebra, $\mathscr{A}_\theta$.

The irrational rotation algebra

But if we take w* closures:

$$\overline{((\tilde{\pi} \times \Lambda)(\mathscr{A}_\theta))}^{w^*} = L^\infty(\mathbb{T})\bar{\rtimes}_\alpha\mathbb{Z}$$

the weak-* crossed product, which we have seen is a type II_1 factor.

On the other hand

$$\overline{((\pi \times \lambda)(\mathscr{A}_\theta))}^{w^*} = \mathscr{B}(L^2(\mathbb{T}))$$

(because $\pi \times \lambda$ is irreducible - ergodicity) so we get a type I_1 factor.

These two von Neumann algebras cannot be isomorphic (not even algebraically) for example because in $\mathscr{B}(L^2(\mathbb{T}))$ the unilateral shift S satisfies $S^*S = I \neq SS^*$ whereas in $L^\infty(\mathbb{T})\bar{\rtimes}_\alpha\mathbb{Z}$ the relation $s^*s = I$ implies $ss^* = I$.

Semicrossed products

Generalisations:

• $\mathscr{A}$ is now an operator algebra (preferably unital), i.e. a norm closed subalgebra of a C*-algebra, not necessarily selfadjoint (for example, the upper triangular matrices on ℓ^2 or the disk algebra $A(\mathbb{D})$).

• G is replaced by a unital sub-semigroup G^+ of a group G (preferably abelian)

• the action α is now a homomorphism $\alpha : G^+ \to \mathrm{End}(\mathscr{A})$ where $\mathrm{End}(\mathscr{A})$ consists of all homomorphisms $\mathscr{A} \to \mathscr{A}$ which are *completely contractive*.

(On a C*-algebra, every *-homomorphism is completely contractive)

The triple $(\mathscr{A}, \alpha, G^+)$ is called a semigroup dynamical system.

Semicrossed products

Restrict to abelian G.
A covariant representation $(\pi, T; H)$ of $(\mathscr{A}, \alpha, G^+)$ is:

$$\begin{aligned} &\pi : \mathscr{A} \to \mathscr{B}(H) \quad \text{compl. contractive representation} \\ &T : G^+ \to \mathscr{B}(H) \quad \text{contactions s.t. } T_{s+t} = T_s T_t. \\ &\pi(f) T_s = T_s \pi(\alpha_s(f)), \quad f \in \mathscr{A}, s \in G^+ \quad \text{(covariance).} \end{aligned}$$

The covariance algebra $c_{00}(G^+, \alpha, \mathscr{A})$ is $c_{00}(G^+) \otimes \mathscr{A}$ as a linear space with

$$(\delta_t \otimes f) * (\delta_s \otimes g) = \delta_{t+s} \otimes \alpha_s(f) g.$$

To define a norm[1], fix a family $\mathscr{F}$ of covariant pairs and put

$$\left\| \sum_k \delta_{t_k} \otimes f_k \right\|_{\mathscr{F}} := \sup \left\{ \left\| \sum_k T_{t_k} \pi(f_k) \right\|_{\mathscr{B}(H)} : (\pi, T : H) \in \mathscr{F} \right\}$$

[1] on the quotient by $\ker \|\cdot\|_{\mathscr{F}}$, if necessary

Semicrossed products

To get an *operator algebra structure* need norms on $n \times n$ matrices for all $n \in \mathbb{N}$: Given $F_k = [f_{i,j}^{(k)}] \in M_n(\mathscr{A})$, for each covariant rep. $(\pi, T : H)$ get operator $[T_{t_k}\pi(f_{i,j}^{(k)})]$ on H^n. Define

$$\left\|\sum_k \delta_{t_k} \otimes F_k\right\|_{n,\mathscr{F}} := \sup\left\{\left\|\sum_k [T_{t_k}\pi(f_{i,j}^{(k)})]\right\|_{\mathscr{B}(H^n)} : (\pi, T : H) \in \mathscr{F}\right\}$$

Definition

The semicrossed product $\mathscr{A} \times_\alpha G^+$ is the Hausdorff[2] completion of $c_{00}(G^+, \alpha, \mathscr{A})$ with respect to $\|\cdot\|_{\mathscr{F}^c}$ where $\mathscr{F}^c$ denotes the family of all contractive covariant pairs.

When one restricts to the family $\mathscr{F}^{is}$ of all isometric covariant pairs, one obtains the isometric semicrossed product $\mathscr{A} \times_\alpha^{is} G^+$.

[2] i.e. the completion of the quotient modulo the ideal $\ker \|\cdot\|_{\mathscr{F}^c}$

Example: the irrational rotation

As before, fix $\theta \in \mathbb{R}$ s.t. $\frac{\theta}{2\pi}$ is irrational. Let $\mathscr{A} = C(\mathbb{T})$, $G = \mathbb{Z}$ and

$$(\alpha_n f)(z) = f(e^{in\theta} z) \quad (f \in \mathscr{A}, n \in \mathbb{Z}, z \in \mathbb{T}).$$

The semicrossed product $C(\mathbb{T}) \rtimes_\alpha \mathbb{Z}_+$ is a closed subalgebra of the irrational rotation algebra $C(\mathbb{T}) \rtimes_\alpha \mathbb{Z}$ (why?).
Thus the representation $\pi \times \lambda : C(\mathbb{T}) \rtimes_\alpha \mathbb{Z} \to \mathscr{B}(L^2(\mathbb{T}))$ restricts to an isometric representation of $C(\mathbb{T}) \rtimes_\alpha \mathbb{Z}_+$ given by (flip)

$$(\pi \times \lambda)(\sum_{k=0}^{n} \delta_k \otimes f_k) = \sum_{k=0}^{n} V^k \pi(f_k)$$

where V is the generator λ_1 of $\{\lambda_n : n \in \mathbb{Z}_+\}$ given by $(Vg)(z) = g(e^{i\theta} z)$, $g \in L^2(\mathbb{T})$.

Example: the irrational rotation

The C*-algebra $C(\mathbb{T})$ is the closed algebra generated by ζ and $\bar{\zeta}$, where $\zeta(z) = z$; hence $\pi(C(\mathbb{T}) \subseteq \mathscr{B}(L^2(\mathbb{T}))$ is generated by $U := \pi(\zeta)$ and U^*. Therefore $C(\mathbb{T}) \rtimes_\alpha \mathbb{Z}_+$ is generated by $\{U, U^*, V\}$ and $C(\mathbb{T}) \rtimes_\alpha \mathbb{Z} = \mathscr{A}_\theta$ is generated by $\{U, U^*, V, V^*\}$.

$$UV = e^{i\theta} VU \quad \text{the Weyl relation.}$$

Proposition

The w-closed subalgebra of $\mathscr{B}(L^2(\mathbb{T})$ generated by $\{U, V\}$ is the nest algebra $\mathrm{Alg}\mathscr{N}$ of all operators $T \in \mathscr{B}(L^2(\mathbb{T}))$ leaving all elements of $\mathscr{N} = \{N_n : n \in \mathbb{Z}\}$ invariant, where $N_n = \{f \in L^2(\mathbb{T}) : \hat{f}(k) = 0, k < n\}$.*

Example: the irrational rotation

After Fourier transform $L^2(\mathbb{T}) \to \ell^2(\mathbb{Z})$:

$$U \sim \begin{bmatrix} \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \\ \dots & 0 & 0 & 0 & 0 & 0 & \dots \\ \dots & 1 & 0 & 0 & 0 & 0 & \dots \\ \dots & 0 & 1 & \mathbf{0} & 0 & 0 & \dots \\ \dots & 0 & 0 & 1 & 0 & 0 & \dots \\ \dots & 0 & 0 & 0 & 1 & 0 & \dots \\ & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}, \; V \sim \begin{bmatrix} \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \\ \dots & \bar{\lambda}^2 & 0 & 0 & 0 & 0 & \dots \\ \dots & 0 & \bar{\lambda} & 0 & 0 & 0 & \dots \\ \dots & 0 & 0 & \mathbf{1} & 0 & 0 & \dots \\ \dots & 0 & 0 & 0 & \lambda & 0 & \dots \\ \dots & 0 & 0 & 0 & 0 & \lambda^2 & \dots \\ & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

Write A_θ^+ and A_θ^{++} for the norm-closed subalgebras of A_θ generated by $\{U, V, V^*\}$ and $\{U, V, I\}$ respectively.

Example: the irrational rotation

Note that $U(N_m) = N_{m+1} \subset N_m$ and $V(N_m) = N_m$.
It follows that U, V and V^* lie in the nest algebra $Alg\mathcal{N}$ and so

$$A_\theta^{++} \subset A_\theta^{+} \subseteq A_\theta \cap Alg\mathcal{N}.$$

We have shown that the weak-* closure of A_θ^{++} is the whole of $Alg\mathcal{N}$. Thus

$$W^*(A_\theta^{++}) = W^*(A_\theta^{+}) = Alg\mathcal{N}.$$

Since A_θ is an irreducible C*-algebra, its w* closure is $B(H)$.

Proposition

We have $A_\theta^{+} = A_\theta \cap Alg\mathcal{N}$. In other words A_θ^{+} is a nest subalgebra of a C-algebra.*