Failure of Spectral Synthesis: The example of Laurent Schwartz

C.R. Acad. Sci. Paris 227, 424–426 (1948)

Let[1]

$$\mathcal{A} = A(\mathbb{R}^3) := \{f \in C_0(\mathbb{R}^3) : \exists \phi \in L^1(\mathbb{R}^3) \text{ s.t. } f = \hat{\phi}\}, \quad \|f\|_A = \|\phi\|_1 .$$

Here $\hat{\phi}(\mathbf{y}) = \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} \phi(\mathbf{x}) \exp(-i\mathbf{x} \cdot \mathbf{y}) dm(\mathbf{x})$

where $\mathbf{x} \cdot \mathbf{y} = x_1y_1 + x_2y_2 + x_3y_3$ and m is Lebesgue measure on $\mathbb{R}^3$. Schwartz showed that the unit sphere in $\mathbb{R}^3$,

$$E = \{\mathbf{y} \in \mathbb{R}^3 : |\mathbf{y}| := (y_1^2 + y_2^2 + y_3^2)^{1/2} = 1\}$$

does not satisfy synthesis, i.e. that $J(E) \subsetneq I(E)$.

To prove this, we shall construct a linear functional ρ on $\mathcal{A}$, continuous for $\|\cdot\|_A$, which annihilates all of $J(E)$ but not $I(E)$.

Define

$$\rho_0 : C_c^\infty(\mathbb{R}^3) \to \mathbb{C} : f \to \int_E \frac{\partial f}{\partial y_3} d\sigma$$

where σ is Lebesgue measure on the sphere E.

The space $C_c^\infty(\mathbb{R}^3)$ of infinitely differentiable functions of compact support is a dense subset of $(A(\mathbb{R}^3), \|\cdot\|_A)$ (exercise). We show that the functional ρ_0 is $\|\cdot\|_A$-continuous:

By Plancherel's formula,

$$\int \hat{h}(\mathbf{y}) d\sigma(\mathbf{y}) = \frac{1}{(2\pi)^{3/2}} \int h(\mathbf{x}) \hat{\sigma}(\mathbf{x}) dm(\mathbf{x})$$

when h and $\hat{h}$ are in $L^1(\mathbb{R}^3)$. Also recall that $D_3\mathcal{F} = -i\mathcal{F}M_3$, so $\frac{\partial \hat{\phi}}{\partial y_3} = \hat{\psi}$ where $|\psi(\mathbf{x})| = |x_3\phi(\mathbf{x})|$. Thus if $f = \hat{\phi} \in C_c^\infty(\mathbb{R}^3)$ then

$$\begin{aligned} |\rho_0(f)| &= \left| \int_E \frac{\partial f}{\partial y_3} d\sigma \right| = \left| \int_E \hat{\psi} d\sigma \right| \le \frac{1}{(2\pi)^{3/2}} \int |\psi(\mathbf{x}) \hat{\sigma}(\mathbf{x})| dm(\mathbf{x}) \\ &= \frac{1}{(2\pi)^3} \int |x_3 \phi(\mathbf{x}) \hat{\sigma}(\mathbf{x})| dm(\mathbf{x}) \\ &\le \frac{1}{(2\pi)^{3/2}} \|\phi\|_1 \sup\{|x_3 \hat{\sigma}(\mathbf{x})| : \mathbf{x} \in \mathbb{R}^3\} \end{aligned}$$

and so it suffices to prove that $\sup\{|x_3\hat{\sigma}(\mathbf{x})| : \mathbf{x} \in \mathbb{R}^3\} < \infty$.

Recall that

$$\hat{\sigma}(\mathbf{x}) = \int_E \exp(-i\mathbf{x} \cdot \mathbf{y}) d\sigma(\mathbf{y}).$$

To estimate this integral, fix a nonzero $\mathbf{x} \in \mathbb{R}^3$ of length a and choose an orthonormal basis $\mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3}$ with $\mathbf{e}_3 = \frac{1}{a}\mathbf{x}$. (Observe that a change of basis will not affect the integral, since the integration is over a spherically symmetric domain). Introducing spherical coordinates $y_1 = r\cos\theta\sin\varphi, y_2 =$

[1] schwartz, Mar 23, 2014

$r\sin\theta\sin\varphi, y_3 = r\cos\varphi$ where $r = 1$ (integration is on E), $\theta \in [0, 2\pi]$ and $\varphi \in [0, \pi]$, we have $\mathbf{x}\cdot\mathbf{y} = |x|y_3$ and so

$$\hat{\sigma}(\mathbf{x}) = \int_0^{\pi}\int_0^{2\pi} \exp(-ia\cos\varphi)\sin\varphi d\theta d\varphi = 2\pi\int_0^{\pi} \exp(-ia\cos\varphi)\sin\varphi d\varphi = 4\pi\frac{\sin a}{a}$$

(indeed $\dfrac{de^{-ia\cos\varphi}}{d\varphi} = e^{-ia\cos\varphi}(ia\sin\varphi)$ etc.). Thus we have shown that

$$|\mathbf{x}||\hat{\sigma}(\mathbf{x})| \leq 4\pi$$

for all $\mathbf{x} \in \mathbb{R}^3$ and hence $|\rho_0(f)| \leq \sqrt{\frac{2}{\pi}}\,\|\phi\|_1 = \sqrt{\frac{2}{\pi}}\,\|f\|_A$.

It therefore follows that the functional ρ_0 extends to a continuous linear functional ρ on $(A(\mathbb{R}^3), \|\cdot\|_A)$.

Now ρ vanishes on $J(E)$. Indeed if $f \in \mathcal{A}$ vanishes in an open neighbourhood of E, it can be approximated, in the norm $\|\cdot\|_A$, by a sequence (f_n) of elements of $C_c^\infty(\mathbb{R}^3)$ that also vanish in an open neighbourhood of E, so that $\frac{\partial f_n}{\partial y_3}$ also vanishes there and so $\rho(f_n) = \rho_0(f_n) = 0$, hence $\rho(f) = 0$.

On the other hand, ρ does not annihilate $I(E)$. Indeed, take any function $g \in C_c^\infty(\mathbb{R})$ with $g(1) = 0$ and $g'(1) \neq 0$ and define $f(\mathbf{y}) = y_3 g(|\mathbf{y}|^2)$. This is in $C_c^\infty(\mathbb{R}^3)$, so $f \in \mathcal{A}$. When $y \in E$, $f(\mathbf{y}) = y_3 g(1) = 0$. Thus $f \in I(E)$. But

$$\frac{\partial f}{\partial y_3} = g(|\mathbf{y}|^2) + y_3 g'(\mathbf{y})2y_3 = 2y_3^2 g'(1)$$

and therefore

$$\rho(f) = \int_E \frac{\partial f}{\partial y_3} d\sigma = 2g'(1)\int_0^{\pi}\int_0^{2\pi} \cos^2\varphi\sin\varphi d\theta d\varphi = 4\pi g'(1)\int_0^{\pi} \cos^2\varphi\sin\varphi d\varphi = \frac{4}{3}\pi g'(1) \neq 0.$$