Lecture of Feb. 21: Summary

Definition 1 (Loginov-Shulman [17], Erdos [?])
(i) The **reflexive cover** $\mathrm{Ref}(\mathcal{S})$ *of a subset* $\mathcal{S} \subseteq \mathcal{B}(\mathcal{H},\mathcal{K})$ *is the set of all* $B \in \mathcal{B}(\mathcal{H},\mathcal{K})$ *such that*

$$Bx \in \overline{[\mathcal{S}x]} \quad \forall x \in \mathcal{H}.$$

(ii) A subset $\mathcal{S} \subseteq \mathcal{B}(\mathcal{H},\mathcal{K})$ *is said to be* **reflexive** *if* $\mathcal{S} = \mathrm{Ref}(\mathcal{S})$.

The reflexive cover of a set of operators can be thought of as its "one-point closure". Of course Ref is not a closure operator in the topological sense. However, in some important cases, the reflexive cover of a subspace coincides with its closure in the weak operator topology (WOT).

Proposition 1 *Let* $\mathcal{S} \subseteq \mathcal{B}(\mathcal{H},\mathcal{K})$ *be a linear space. Then*

$$\overline{\mathcal{S}}^{\mathrm{sot}} = \{T \in \mathcal{B}(\mathcal{H},\mathcal{K}) : \forall n,\ T^{(n)} \in \mathrm{Ref}(\mathcal{S}^{(n)}\}.$$

Corollary 2 *Let* $\mathcal{S} \subseteq \mathcal{B}(\mathcal{H},\mathcal{K})$ *be a linear space. Then* $\overline{\mathcal{S}}^{\mathrm{sot}} = \overline{\mathcal{S}}^{\mathrm{wot}}$.

(In fact, this is true for any *convex* set $\mathcal{S}$.)

Example If

$$\mathcal{A} = \left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} : a, b \in \mathbb{C} \right\}$$

then $\mathcal{A}$ is not reflexive: $\mathrm{Ref}(\mathcal{A})$ is the set of all upper-triangular matrices.

Theorem 3 (von Neumann's bicommutant theorem) *Let* $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H})$ *be a selfadjoint algebra containing the identity operator. Then*

$$\overline{\mathcal{A}}^{\mathrm{wot}} = \mathcal{A}''.$$

Remark 4 *In fact,* $\mathcal{A}'' = \mathrm{Ref}(\mathcal{A})$ *in this case. Thus a topological property* $\mathcal{A} = \overline{\mathcal{A}}^{\mathrm{wot}}$ *is shown to be equivalent to an algebraic property (* $\mathcal{A} = \mathcal{A}''$*) and also to a 'geometric' property (*$\mathcal{A} = \mathrm{Ref}(\mathcal{A})$*) (in the sense that it relates to the action of* $\mathcal{A}$ *on the Hilbert space).*

A crucial observation is that reflexive subspaces can be characterised in terms of rank one operators[1]. Indeed,

$$\begin{aligned}
\mathrm{Ref}(\mathcal{S}) &= \{T \in \mathcal{B}(\mathcal{H},\mathcal{K}) : \langle \mathcal{S}x, y\rangle = 0 \Rightarrow \langle Tx, y\rangle = 0\} \\
&= \{T \in \mathcal{B}(\mathcal{H},\mathcal{K}) : \omega_{x,y} \perp \mathcal{S} \Rightarrow \omega_{x,y} \perp T\} \\
&= (\mathcal{R}_1({}^{\perp}\mathcal{S}))^{\perp} \qquad (1)
\end{aligned}$$

[1] This is due to Larson [?] in the case of unital algebras, and to Kraus-Larson [?] and Erdos [?] in the general case. Theorem *5* is from [?], Theorem 9.2.

where $\mathcal{R}_1(\mathcal{T})$ denotes the 'rank one subspace' of $\mathcal{T}$ (the linear span of the vector functionals in $\mathcal{T}$) and $\omega_{x,y}(T) = \langle Tx, y\rangle$.

Thus reflexive spaces are (post-) annihilators of sets of rank ones. The converse also holds. Thus

Proposition 5 *A set $\mathcal{S} \subseteq \mathcal{B}(\mathcal{H}, \mathcal{K})$ is reflexive if and only if it is of the form $\mathcal{S} = \mathcal{R}^{\perp}$, for some set $\mathcal{R} \subseteq \mathcal{B}(\mathcal{K}, \mathcal{H})$ of rank one operators.*

Proof Let $\mathcal{S} = \mathcal{R}^{\perp}$. Suppose that $Tx \in \overline{\mathcal{S}x}$ for all $x \in H$. Then for each $\omega_{x,y} \in \mathcal{R}$, we have $\omega_{x,y} \perp \mathcal{S}$, i.e. $\langle \mathcal{S}x, y\rangle = \{0\}$ and hence $\langle Tx, y\rangle = 0$. Thus $T \in \mathcal{R}^{\perp} = \mathcal{S}$. $\square$

Reflexive masa bimodules Let H, K be Hilbert spaces, $\mathcal{A} \subseteq \mathcal{B}(H)$ and $\mathcal{B} \subseteq \mathcal{B}(K)$ masas. A linear subspace $\mathcal{S} \subseteq \mathcal{B}(H, K)$ is said to be an $(\mathcal{A}, \mathcal{B})$-bimodule when $\mathcal{A}\mathcal{S}\mathcal{B} \subseteq \mathcal{S}$, i.e. when $A \in \mathcal{A}$, $B \in \mathcal{B}$ and $S \in \mathcal{S}$ imply $ASB \in \mathcal{S}$.

Proposition 6 *Let $H = L^2(X, \mu)$, $K = L^2(Y, \nu)$ and consider the masas $\mathcal{M}_\mu$ and $\mathcal{M}_\nu$. For any $\Omega \subseteq X \times Y$ the space*

$$\mathfrak{M}_{\max}(\Omega) := \{T \in \mathcal{B}(H, K) : T \textit{ is supported in } \Omega\}$$

is a reflexive masa bimodule.

In fact, all reflexive masa bimodules are of this form:

Recall that any separable acting masa is unitarily equivalent to the multiplication masa $\mathcal{M}_\mu$ of a measure space (X, μ) and that in fact there exists a topology making X a compact metric space and μ a regular Borel measure.

Theorem 7 *[7, 4.2] Let $\mathcal{A} \subseteq \mathcal{B}(H)$ and $\mathcal{B} \subseteq \mathcal{B}(K)$ be separably acting masas. Suppose $(\mathcal{A}, H) \stackrel{u}{\simeq} (\mathcal{M}_\mu, L^2(X, \mu))$ and $(\mathcal{B}, K) \stackrel{u}{\simeq} (\mathcal{M}_\nu, L^2(Y, \nu))$. If $\mathcal{S} \subseteq \mathcal{B}(H, K)$ is a reflexive $(\mathcal{A}, \mathcal{B})$-bimodule, then there exists a subset $\Omega \subseteq X \times Y$ such that*

$$\mathcal{S} \stackrel{u}{\simeq} \mathfrak{M}_{\max}(\Omega).$$

The proof uses the following:

Proposition 8 *[7, 3.4] Let (X, μ) and (Y, ν) be compact spaces equipped with regular Borel measures. If $K \subseteq X \times Y$ is ω-closed and*

$$K \subseteq \bigcup_{n=1}^{\infty} A_n \times B_n$$

where $A_n \subseteq X$ and $B_n \subseteq Y$ are Borel sets, then for all $\epsilon > 0$ there exists $X_\epsilon \subseteq X$, $Y_\epsilon \subseteq Y$ with $\mu(X \setminus X_\epsilon) < \epsilon$ and $\nu(Y \setminus Y_\epsilon) < \epsilon$ and $N \in \mathbb{N}$ so that

$$K \cap (X_\epsilon \times Y_\epsilon) \subseteq \bigcup_{n=1}^{N} A_n \times B_n.$$

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