Lectures of March 7 and 14: Some notes

Preliminaries Let $(\mathcal{A}, \|\cdot\|)$ be[1] a commutative Banach algebra and let $\sigma(\mathcal{A})$ be its *spectrum*, namely the set of all nonzero homomorphisms $\phi : \mathcal{A} \to \mathbb{C}$. Equipped with the topology of pointwise convergence, $\sigma(\mathcal{A})$ is a locally compact Hausdorff space and each $a \in \mathcal{A}$ defines a continuous function $\hat{a} : \sigma(\mathcal{A}) \to \mathbb{C}$ by $\hat{a}(\phi) := \phi(a)$. The algebra $\mathcal{A}$ is *semisimple* if for each nonzero $a \in \mathcal{A}$ there exists $\phi \in \sigma(\mathcal{A})$ such that $\phi(a) \neq 0$, i.e. if the map $a \to \hat{a}$ is one to one. In this case (by identifying a with $\hat{a}$) $\mathcal{A}$ can and will be identified with a subalgebra of the algebra $C_0(X)$ of continuous complex-valued functions on the spectrum $X = \sigma(\mathcal{A})$ vanishing at infinity, and $\|a\|_\infty \leq \|a\|$ for all $a \in \mathcal{A}$.

A semisimple commutative Banach algebra $\mathcal{A}$ with spectrum X is called *regular* if for each $x \in X$ and $E \subseteq X$ closed with $x \notin E$ there exists $a \in \mathcal{A}$ such that $a(x) = 1$ and $a|_E = 0$.

For example, the algebra of all continuous functions on the closed complex disc $\overline{\mathbb{D}}$ which are holomorphic in the open disc is a semisimple Banach algebra in the sup norm (its spectrum is actually homeomorphic to $\overline{\mathbb{D}}$) but it is *not regular*, because any holomorphic function vanishing on a subset of $\mathbb{D}$ with nonempty interior must vanish everywhere by the identity principle.

Example *Let $A(\mathbb{T})$ (the Wiener or Fourier algebra of the group $\mathbb{T} = \{e^{it} : t \in \mathbb{R}\}$) be the set of all continuous functions $f : \mathbb{T} \to \mathbb{C}$ whose Fourier series $\sum \hat{f}(n)e^{int}$ converges absolutely. With the norm $\|f\|_A := \|\hat{f}\|_1 = \sum |\hat{f}(n)|$ and pointwise operations, $A(\mathbb{T})$ is a semisimple Banach algebra (its spectrum is actually homeomorphic to $\mathbb{T}$) which is regular.*

In fact it has the following formally [2] stronger property: the singleton $\{x\}$ can be replaced by a compact set. Indeed

Lemma 1 *If K, E are disjoint compact subsets of $\mathbb{T}$, there exists $f \in A(\mathbb{T})$ such that $f|_K = 1$ and $f|_E = 0$.* [3]

Proof For $\epsilon > 0$, let $K_\epsilon = \{e^{it} \in \mathbb{T} : |t - s| < \epsilon \text{ for some } e^{is} \in K\}$. Choose $\epsilon > 0$ small enough so that $K_{2\epsilon} \cap E = \emptyset$. If m denotes normalised Lebesgue measure on $\mathbb{T}$ and $V = \{e^{is} : |s| < \epsilon\}$, we set

$$f(e^{it}) = \frac{1}{\epsilon} \int \chi_{K_\epsilon}(e^{is}) \chi_V(e^{i(t-s)}) \frac{ds}{2\pi} = \frac{m(e^{it}V \cap K_\epsilon)}{\epsilon}$$

(this equality holds because $e^{i(t-s)} \in V \iff e^{-is} \in e^{-it}V \iff e^{is} \in e^{it}V$).

Let $e^{it} \in K$. If $|s| < \epsilon$ then $e^{i(t+s)} \in K_\epsilon$; thus $e^{it}V \subseteq K_\epsilon$ and so $m(e^{it}V \cap K_\epsilon) = m(e^{it}V) = \epsilon$; hence $f(e^{it}) = 1$.

On the other hand, if $e^{it} \in E$ then $e^{it} \notin K_{2\epsilon}$ and so $e^{it}V \cap K_\epsilon = \emptyset$; [4] thus $f(e^{it}) = 0$.

It remains to prove that f is indeed in $A(\mathbb{T})$, i.e. that its Fourier transform is in $\ell^1(\mathbb{Z})$. For this, notice that f is the convolution of two functions which are both in $L^2(\mathbb{T})$: $f = \frac{1}{\epsilon}\chi_{K_\epsilon} * \chi_V$; therefore, applying the Fourier transform $\mathcal{F}$,

$$\mathcal{F}(f) = \frac{1}{\epsilon}\mathcal{F}(\chi_{K_\epsilon} * \chi_V) = \mathcal{F}(\chi_{K_\epsilon}) \cdot \mathcal{F}(\chi_V).$$

[1] analy2, 12Mar2014 revised 16Mar, 21Mar

[2] It can actually be shown that any commutative semisimple regular Banach algebra has the property that a compact set can be separated from a disjoint closed set by an element of the algebra.

[3] This Lemma actually holds for the Fourier algebra $A(G)$ of any locally compact group, with essentially the same ideas for the proof (provided one defines $A(G)$ appropriately).

[4] For if $e^{iu} \in e^{it}V \cap K_\epsilon$ then $|u - t| < \epsilon$ and also $|u - s| < \epsilon$ for some $s \in K$ and so $|t - s| < 2\epsilon$ which gives $e^{it} \in K_{2\epsilon}$

Now we know (!) that the Fourier transform of a function in $L^2(\mathbb{T})$ is in fact in $\ell^2(\mathbb{Z})$. Hence $\mathcal{F}(f)$ is the product of two ℓ^2 sequences, and thus is in ℓ^1 (Cauchy-Schwarz). $\square$

The extremal ideals Let $(\mathcal{A}, \|\cdot\|)$ be a commutative semisimple and regular Banach algebra with spectrum X and let $E \subseteq X$ be a closed subset. Define

$$\begin{aligned} I(E) &= \{f \in \mathcal{A} : f|_E = 0\} \\ J_0(E) &= \{f \in \mathcal{A} : \operatorname{supp} f \text{ compact and } \operatorname{supp} f \cap E = \emptyset\} \\ J(E) &= \overline{J_0(E)}. \end{aligned}$$

Note that any $f \in J_0(E)$ vanishes (not only on E, but) in the open neighbourhood $(\operatorname{supp} f)^c$ of E. Recall that $Z(J) = \{x \in X : f(x) = 0 \text{ for all } f \in J\}$.

We will prove the following Theorem:

Theorem 2 *Let $J \subseteq \mathcal{A}$ be a closed ideal and $E \subseteq X$ a closed subset. Then*

$$Z(J) = E \quad \textit{if and only if} \quad J_0(E) \subseteq J \subseteq I(E).$$

In particular, $J(E)$ is the smallest closed ideal J with null set $Z(J) = E$

We will need some preliminaries.

Proposition 3 *If $b \in \mathcal{A}$ vanishes nowhere on a compact set $F \subseteq X$, there is a function d* contained in $\mathcal{A}$ *such that $(bd)(t) = 1$ for all $t \in F$.*

Proof Consider the commutative Banach algebra $\mathcal{B} = \mathcal{A}/I(F)$ and let $q : \mathcal{A} \to \mathcal{B}$ be the quotient map. I claim that the character space $\sigma(\mathcal{B})$ is homeomorphic to F:

For any point $x \in F$, the associated multiplicative linear functional $a \to a(x)$ annihilates $I(F)$, hence induces a multiplicative linear functional $\phi_x : \mathcal{B} \to \mathbb{C}$ by $\phi_x(q(a)) = a(x)$ (this ϕ_x is well defined because if $q(a) = q(b)$ then $a - b \in I(F)$ and so $a(x) = b(x)$). Conversely, every nonzero multiplicative linear functional $\phi : \mathcal{B} \to \mathbb{C}$ defines a multiplicative linear functional $\phi \circ q : \mathcal{A} \to \mathbb{C}$ which is nonzero because ϕ is nonzero and q is onto. Therefore there exists $x \in X$ such that $\phi(q(a)) = a(x)$ for all $a \in \mathcal{A}$ and since $\phi \circ q$ annihilates $I(F)$, x must lie in $Z(I(F)) = F$; thus $\phi \circ q = \phi_x$.

Hence the map $F \to \sigma(\mathcal{B}) : x \to \phi_x$ is a bijection. Finally, if $x_i \to x$ in X then $a(x_i) \to a(x)$ for all $a \in \mathcal{A}$ (definition of the topology of $X = \sigma(\mathcal{A})$) hence $\phi_{x_i}(q(a)) \to \phi_x(q(a))$, which is equivalent to $\phi_{x_i} \to \phi_x$ in the topology of $\sigma(\mathcal{B})$. This shows that $x \to \phi_x$ is continuous.

Thus $\sigma(\mathcal{B})$ is compact and $x \to \phi_x$ a homeomorphism.

Note that $\mathcal{B} = \mathcal{A}/I(F)$ is semisimple: indeed if $q(a) \neq 0$ then there exists $t \in F$ s.t. $a(t) \neq 0$; thus $\phi_t(q(a)) = a(t) \neq 0$.

The fact that $\mathcal{B}$ has compact spectrum and contains a nowhere vanishing element (namely, $q(b)$) implies that $\mathcal{B}$ *has a unit, say e*. Equivalently

there exists an element $u \in \mathcal{A}$ such that $q(u) = e$ i.e. $u|_F = 1$. [5] We will prove this in Proposition 6 below.

[5] indeed, for all $x \in F$, $u(x) = \phi_x(q(u)) = \phi_x(e) = 1$.

I claim that $q(b)$ is invertible in $\mathcal{B}$. This is equivalent to showing that $\phi(q(b)) \neq 0$ for every nonzero multiplicative linear functional $\phi \in \sigma(B)$. But as observed above, any such ϕ must be of the form ϕ_x for some $x \in X$. Thus $\phi_x(q(b)) = b(x)$ which is never zero because b never vanishes on F. Thus $q(b)$ is invertible.

It follows that there exists $q(d) = (q(b))^{-1} \in \mathcal{B}$ such that $q(d)q(u) = q(u)$; but this means exactly that $q(du - u) = 0$ in $\mathcal{B}$, i.e. that $(db)(t) = u(t) = 1$ for all $t \in F$, as required. □

Remark 4 *Note that in many specific cases the existence of the element u is proved directly. This is the case for example in the case of the Wiener algebra $A(\mathbb{T})$ (see Lemma 1) or more generally for $A(G)$. A proof in the general case (Proposition 6) seems to require some complex analysis.*

Proposition 5 *Let $J \subseteq \mathcal{A}$ be a closed ideal and $K \subseteq X$ a compact set such that $Z(J) \cap K = \emptyset$. Then there exists $c \in J$ such that $c(x) = 1$ for all $x \in K$.*

Proof For all $x \in K$, since $x \notin Z(J)$ there is $b \in J$ with $b(x) \neq 0$. Since b is continuous, there is an open neighbourhood V_x of x on which b never vanishes and such that $\overline{V}_x$ is compact.[6]

Since b never vanishes on the compact set $\overline{V}_x$, by Proposition 3 there is a function d *contained in* $\mathcal{A}$ such that $(bd)(t) = 1$ for all $t \in \overline{V}_x$.

Set $c_x = bd$. Thus $c_x \in J$ because J is an ideal and $c_x(t) = 1$ for all $t \in \overline{V}_x$. The family $\{V_x : x \in K\}$ is an open cover of K. Let $V_1, \ldots, V_n$ be a finite subcover and denote by $c_1, \ldots, c_n$ the corresponding elements of J. If we now define

$$c = \mathbf{1} - \prod_{i=1}^{n} (\mathbf{1} - c_i)$$

then we have an element of J (the $\mathbf{1}$'s cancel out) such that $c(x) = 1$ for any $x \in K$ (because x will be in some V_i so $(\mathbf{1} - c_i)(x) = 0$ so the product vanishes and hence $c(x) = 1$. □

Proof of Theorem 2 Suppose first that $J_0(E) \subseteq J \subseteq I(E)$; to show that $Z(J) = E$. Now $Z(J_0(E)) \supseteq Z(J) \supseteq Z(I(E))$. But we know (regularity) that $Z(I(E)) = E$. Thus it suffices to prove that $E \supseteq Z(J_0(E))$.

So let $x \notin E$. Then x has an open neighbourhood U s.t. $\overline{U}$ is compact and disjoint from E. By regularity, there exists $a \in \mathcal{A}$ with $a(x) = 1$ and $a(t) = 0$ for all $t \in U^c$ (a closed set not containing x). Thus $\{t \in X : a(t) \neq 0\} \subseteq U$ and so $\operatorname{supp} a \subseteq \overline{U}$. Thus $\operatorname{supp} a$ is compact and does not meet E; hence $a \in J_0(E)$. Since $a(x) \neq 0$, we have shown that $x \notin Z(J_0(E))$.

Suppose now that $Z(J) = E$. It is obvious that $J \subseteq I(E)$. To show that $J_0(E) \subseteq J$, take any $a \in J_0(E)$ and put $K = \operatorname{supp} a$: a compact set with $K \cap Z(J) = \emptyset$.

By Proposition 5, there exists $c \in J$ such that $c(x) = 1$ for all $x \in K$.

But note that $a = ac$; indeed $(ac)(x) = a(x)$ for all $x \in K$ and $(ac)(y) = 0 = a(y)$ for all $y \notin K$. Thus $a \in J$ since $c \in J$ and J is an ideal. This completes the proof that $J_0(E) \subseteq J$. □

Proposition 6 *If $\mathcal{B}$ is a semisimple commutative Banach algebra with compact spectrum containing an element a which vanishes nowhere, then $\mathcal{B}$ must be unital, and in fact a must be invertible in $\mathcal{B}$.*

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[6] for instance, take $V_x = \{t \in X : |b(t)| > \frac{1}{2}|b(x)|\}$. Then $\overline{V}_x$ is contained in $\{t \in X : |b(t)| \geq \frac{1}{2}|b(x)|\}$ which is compact because $|b(t)| < \frac{1}{2}|b(x)|$ for all t outside some compact set (recall that $\mathcal{A} \subseteq C_0(X)$).

Proof. Consider a as an element of the unitisation $\mathcal{B}_1 = \mathcal{B} \oplus \mathbb{C}$ and write $\mathbf{e}$ for the unit $(0,1)$ of $\mathcal{B}_1$ so that each $(f,\lambda) \in \mathcal{B}_1$ is written $f + \lambda\mathbf{e}$. The spectrum $\sigma(\mathcal{B}_1)$ consists of all maps $\tilde{\phi} : f + \lambda\mathbf{e} \to \phi(f) + \lambda$ where $\phi \in \sigma(\mathcal{B})$ together with the map $\phi_\infty : f + \lambda\mathbf{e} \to \lambda$. Therefore the spectrum of a in $\mathcal{B}_1$ is the set

$$\{\tilde{\phi}(a) : \phi \in \sigma(\mathcal{B})\} \cup \{\phi_\infty(a)\} = \{\phi(a) : \phi \in \sigma(\mathcal{B})\} \cup \{0\}.$$

Since the map $\hat{a} : \phi \to \phi(a)$ is continuous and never vanishes on $\sigma(\mathcal{B})$ (by definition of the topology of $\sigma(\mathcal{B})$), The set $A := \{\phi(a) : \phi \in \sigma(\mathcal{B})\}$ is a compact subset of $\mathbb{C}$ not containing $0 \in \mathbb{C}$. Thus there are disjoint open subsets V, W of $\mathbb{C}$ such that $0 \in V$ and $A \subseteq W$. Now define a function $h : V \cup W \to \mathbb{C}$ by

$$h(z) = \begin{cases} 0, & z \in V \\ 1/z, & z \in W \end{cases}$$

This function has a complex derivative at all points of $V \cup W$: it is a holomorphic function on the open set $V \cup W$.

Let γ be a closed piecewise smooth cycle (: finite 'sum' of simple closed curves) in $V \cup W$ such that $\mathrm{Ind}_\gamma(z) = 1$ for all $z \in \{0\} \cup A$ and $\mathrm{Ind}_\gamma(z) = 0$ for all $z \notin V \cup W$ (it is proved in complex analysis that such curves exist - see for example Negr. 12.8). Then by the global Cauchy Theorem (Negr. 5.19),

$$h(z) = \frac{1}{2\pi i} \int_\gamma h(w)(z-w)^{-1} dw, \qquad z \in \{0\} \cup A.$$

Since the spectrum of a in $\mathcal{B}_1$ is the set $\{0\} \cup A$, it follows that when $w \in V \cup W \setminus (\{0\} \cup A)$ the element $a - w\mathbf{e}$ is invertible in $\mathcal{B}_1$ and it is known that the map $w \to (a - w\mathbf{e})^{-1}$ is continuous there; in particular, this map is defined and continuous on the trace of γ. It follows that the integral

$$\tilde{h}(a) := \frac{1}{2\pi i} \int_\gamma h(w)(a - w\mathbf{e})^{-1} dw$$

is well defined as a limit of Riemann sums which are elements of $\mathcal{B}_1$. Moreover for every nonzero multiplicative linear functional $\psi : \mathcal{B}_1 \to \mathbb{C}$ we have of course $\psi((a - w\mathbf{e})^{-1}) = (\psi(a) - w\psi(\mathbf{e}))^{-1}) = (\psi(a) - w)^{-1}$ (since $\psi(\mathbf{e}) = 1$) and so by linearity and continuity,

$$\psi(\tilde{h}(a)) = \frac{1}{2\pi i} \int_\gamma h(w)\psi((a - w\mathbf{e})^{-1}) dw = \frac{1}{2\pi i} \int_\gamma h(w)(\psi(a) - w)^{-1} dw = h(\psi(a))$$

using the integral formula for $h(z)$. In particular, $\phi_\infty(\tilde{h}(a)) = h(\phi_\infty(a)) = h(0) = 0$, which shows that $\tilde{h}(a)$ is actually an element v of $\mathcal{B}$. Thus, for every $\phi \in \sigma(\mathcal{B})$ we have

$$\phi(va) = \phi(\tilde{h}(a)) \cdot \phi(a) = h(\phi(a)) \cdot \phi(a).$$

But $\phi(a) \in A$ and $h(z) = 1/z$ for $z \in A$; therefore $h(\phi(a)) \cdot \phi(a) = 1$ and so $\phi(va) = 1$. Let $u = va$. For all $c \in \mathcal{B}$ we have

$$\phi(uc) = \phi(va) \cdot \phi(c) = \phi(c) \quad \text{or} \quad \phi(uc - c) = 0$$

for all $\phi \in \sigma(\mathcal{B})$. Since $\mathcal{B}$ is semisimple, it follows that $uc = c$ for all $c \in \mathcal{B}$, and so u is the identity of $\mathcal{B}$, and a is invertible with inverse v, as required.

Remark 7 *It can be shown that the assumption that $\mathcal{B}$ contains a nowhere vanishing element is superfluous; but the proof uses tools from the theory of several complex variables...*