# The additivity of the index of Fredholm-like linear maps

I. Exact sequences. All vector spaces considered below are over the field of complex numbers. We say that a sequence of vector spaces and linear maps

$$U \xrightarrow{f} V \xrightarrow{g} W$$

is exact if $\operatorname{im} f = \ker g$. In this way, the sequence

$$0 \longrightarrow U \xrightarrow{f} V$$

is exact if and only if $f$ is injective, whereas the sequence

$$U \xrightarrow{f} V \longrightarrow 0$$

is exact if and only if $f$ is surjective. We say that a sequence of vector spaces and linear maps

$$U \xrightarrow{f} V \xrightarrow{g} W \xrightarrow{h} Z \longrightarrow \ldots$$

is exact if $\operatorname{im} f = \ker g, \operatorname{im} g = \ker h, \operatorname{im} h = \ldots$

An exact sequence of the form

$$0 \longrightarrow U \xrightarrow{f} V \xrightarrow{g} W \longrightarrow 0$$

is called short exact. We note that, in this case, $f$ induces an isomorphism between $U$ and the subspace $\operatorname{im} f \subseteq V$, whereas $W$ is isomorphic with the quotient space $V/\operatorname{im} f = \operatorname{coker} f$, in such a way that $g$ is identified with the quotient map $V \longrightarrow \operatorname{coker} f$. It follows that we then have an equality $\dim V = \dim U + \dim W$.

**Proposition 1.** (i) Let $U \xrightarrow{f} V \xrightarrow{g} W$ be an exact sequence and assume that both $U, W$ are finite dimensional. Then, $\dim V < \infty$ as well.

(ii) Let

$$0 \longrightarrow U_0 \longrightarrow U_1 \longrightarrow \ldots \longrightarrow U_{n-1} \longrightarrow U_n \longrightarrow 0$$

be an exact sequence of finite dimensional vector spaces. Then, $\sum_{i=0}^{n} (-1)^i \dim U_i = 0$.

*Proof.* (i) We note that $\dim \operatorname{im} f \leq \dim U$ and $\dim \operatorname{im} g \leq \dim V$. Since $\operatorname{im} f = \ker g$, it follows that $\dim V = \dim \ker g + \dim \operatorname{im} g \leq \dim U + \dim W$.

(ii) The result is obvious if $n = 0$ (the exactness of $0 \longrightarrow U_0 \longrightarrow 0$ implies that $U_0 = 0$) or $n = 1$ (the exactness of $0 \longrightarrow U_0 \xrightarrow{f} U_1 \longrightarrow 0$ implies that $f$ is an isomorphism). We use induction, assuming that $n \geq 2$. We denote by $f$ the map $U_{n-1} \longrightarrow U_n$ and note that there are exact sequences

$$0 \longrightarrow U_0 \longrightarrow U_1 \longrightarrow \ldots \longrightarrow U_{n-2} \longrightarrow \ker f \longrightarrow 0$$

(since $\operatorname{im}(U_{n-2} \longrightarrow U_{n-1}) = \ker f$) and

$$0 \longrightarrow \ker f \longrightarrow U_{n-1} \longrightarrow U_n \longrightarrow 0.$$

In view of the induction hypothesis, we have $\sum_{i=0}^{n-2} (-1)^i \dim U_i + (-1)^{n-1} \dim \ker f = 0$. Since $\dim \ker f = \dim U_{n-1} - \dim U_n$, it follows that

$$\sum_{i=0}^{n} (-1)^i \dim U_i = \sum_{i=0}^{n-2} (-1)^i \dim U_i + (-1)^{n-1} (\dim U_{n-1} - \dim U_n) = 0,$$

as needed. $\square$

II. The snake lemma. We prove the following result by a method which is usually referred to as *diagram chasing.*

**Proposition 2.** Let

$$\begin{array}{ccccccc} & & U & \xrightarrow{f} & V & \xrightarrow{g} & W & \longrightarrow & 0 \\ & & {\scriptstyle a}\downarrow & & {\scriptstyle b}\downarrow & & {\scriptstyle c}\downarrow & & \\ 0 & \longrightarrow & U' & \xrightarrow{\varphi} & V' & \xrightarrow{\gamma} & W' & & \end{array}$$

be a commutative diagram of vector spaces and linear maps with exact rows. Then, there exists a 6-term exact sequence of vector spaces and linear maps

$$\ker a \xrightarrow{f_0} \ker b \xrightarrow{g_0} \ker c \xrightarrow{\delta} \operatorname{coker} a \xrightarrow{\overline{\varphi}} \operatorname{coker} b \xrightarrow{\overline{\gamma}} \operatorname{coker} c,$$

where $f_0$ and $g_0$ are restrictions of $f$ and $g$, whereas $\overline{\varphi}$ and $\overline{\gamma}$ are obtained from $\varphi$ and $\gamma$ by passage to the quotient.

*Proof.* We begin by defining $\delta$. To that end, let $w$ be an element contained in the kernel $\ker c$ of $c$. Since $g$ is onto, we may write $w = g(v)$ for some $v \in V$. Since $\gamma(b(v)) = (\gamma b)(v) = (cg)(v) = c(g(v)) = c(w) = 0 \in W'$, we conclude that $b(v) \in \ker\gamma = \operatorname{im}\varphi$. Hence, there exists an element $u' \in U'$, such that $b(v) = \varphi(u')$. Having chosen another element $v_0 \in V$ with $w = g(v_0)$, we obtain as above an element $u'_0 \in U'$, such that $b(v_0) = \varphi(u'_0)$. Since $g(v) = w = g(v_0)$, we conclude that $g(v - v_0) = 0 \in W$ and hence $v - v_0 \in \ker g = \operatorname{im} f$. It follows that there exists an element $u \in U$, such that $v - v_0 = f(u)$. We then have

$$\varphi(u'-u'_0) = \varphi(u')-\varphi(u'_0) = b(v)-b(v_0) = b(v-v_0) = b(f(u)) = (bf)(u) = (\varphi a)(u) = \varphi(a(u)).$$

Since $\varphi$ is injective, it follows that $u' - u'_0 = a(u) \in \operatorname{im} a$ and hence the classes of $u'$ and $u'_0$ in the quotient $U'/\operatorname{im} a = \operatorname{coker} a$ are equal. We may therefore define $\delta$, by letting $w \mapsto u' + \operatorname{im} a \in \operatorname{coker} a$.

Having defined $\delta$, we shall now verify the exactness of the sequence in the statement of the proposition.

Since $\operatorname{im} f = \ker g$, it follows that $gf = 0$ and hence $g_0 f_0 = 0$; therefore, we have an inclusion $\operatorname{im} f_0 \subseteq \ker g_0$. In order to prove the reverse inclusion, let $v \in \ker b$ be an element contained in the kernel $\ker g_0$ of $g_0$. Then, $v \in \ker g = \operatorname{im} f$ and hence $v = f(u)$ for some $u \in U$. Since $v \in \ker b$, we conclude that $\varphi(a(u)) = (\varphi a)(u) = (bf)(u) = b(f(u)) = b(v) = 0$. In view of the injectivity of $\varphi$, it follows that $a(u) = 0$ and hence $u \in \ker a$. Hence, $v = f_0(u) \in \operatorname{im} f_0$.

If $v \in \ker b$, then in order to compute $\delta(g_0(v))$, we may choose $v \in V$ as a preimage of $g_0(v) = g(v)$ under $g$. Since $b(v) = 0 = a(0)$, we conclude that $\delta(g_0(v)) = 0 + \operatorname{im} f \in \operatorname{coker} a$. It follows that $\delta g_0 = 0$ and hence $\operatorname{im} g_0 \subseteq \ker\delta$. In order to prove the reverse inclusion, let $w \in \ker c$ be an element contained in the kernel $\ker\delta$ of $\delta$. Then, we may write $w = g(v)$ for some $v \in V$ and $b(v) = \varphi(u')$ for some $u' \in U'$, which is contained in the image of $a$ (so that the class $u' + \operatorname{im} a = \delta(w)$ be zero in $\operatorname{coker} a$). If we write $u' = a(u)$, then we have

$$b(v) = \varphi(u') = \varphi(a(u)) = (\varphi a)(u) = (bf)(u) = b(f(u))$$

and hence the element $v - f(u)$ is contained in the kernel of $b$. Since $g_0(v-f(u)) = g(v-f(u)) = g(v) - g(f(u)) = w - 0 = w$, we conclude that $w \in \operatorname{im} g_0$.

Let $w \in \ker c$ and assume that $\delta(w) = u' + \operatorname{im} a \in \operatorname{coker} a$. Then, by the very definition of $\delta$, there exists an element $v \in V$, such that $w = g(v)$ and $b(v) = \varphi(u')$. Hence, we may conclude that $\overline{\varphi}(\delta(w)) = \overline{\varphi}(u' + \operatorname{im} a) = \varphi(u') + \operatorname{im} b = b(v) + \operatorname{im} b = 0 + \operatorname{im} b \in \operatorname{coker} b$. It follows that $\overline{\varphi}\delta = 0$ and hence $\operatorname{im}\delta \subseteq \ker\overline{\varphi}$. In order to show the reverse inclusion, let $u' \in U'$ be an element, such that the class $u' + \operatorname{im} a \in \operatorname{coker} a$ is contained in the kernel $\ker\overline{\varphi}$ of $\overline{\varphi}$. Then, $\varphi(u') + \operatorname{im} b = 0 + \operatorname{im} b \in \operatorname{coker} b$ and hence $\varphi(u') \in \operatorname{im} b$. Therefore, we may write $\varphi(u') = b(v)$ for some $v \in V$. Since $c(g(v)) = (cg)(v) = (\gamma b)(v) = \gamma(b(v)) = \gamma(\varphi(u')) = 0$, we conclude that $g(v) \in \ker c$. In view of the definition of $\delta$, we have $\delta(g(v)) = u' + \operatorname{im} a$ and hence $u' + \operatorname{im} a \in \operatorname{im}\delta$.

Since $\operatorname{im}\varphi = \ker\gamma$, it follows that $\gamma\varphi = 0$ and hence $\overline{\gamma}\,\overline{\varphi} = 0$; therefore, we have an inclusion $\operatorname{im}\overline{\varphi} \subseteq \ker\overline{\gamma}$. In order to prove the reverse inclusion, let $v' \in V'$ be an element, such that the class $v' + \operatorname{im} b \in \operatorname{coker} b$ is contained in the kernel $\ker\overline{\gamma}$ of $\overline{\gamma}$. Then, $\gamma(v') + \operatorname{im} c = 0 + \operatorname{im} c \in \operatorname{coker} c$ and hence $\gamma(v') \in \operatorname{im} c$. We may then write $\gamma(v') = c(w)$ for some $w \in W$. Since $g$ is onto, we may write $w = g(v)$ for some $v \in V$ and hence we have

$$\gamma(v') = c(w) = c(g(v)) = (cg)(v) = (\gamma b)(v) = \gamma(b(v)).$$

It follows that $v' - b(v) \in \ker\gamma = \operatorname{im}\varphi$ and hence we may write $v' - b(v) = \varphi(u')$ for some $u' \in U'$. We conclude that $v' - \varphi(u') = b(v) \in \operatorname{im} b$ and hence $v' + \operatorname{im} b = \varphi(u') + \operatorname{im} b = \overline{\varphi}(u' + \operatorname{im} a) \in \operatorname{im}\overline{\varphi}$. □

**Corollary 3.** Let $U, V, W$ be vector spaces and consider two linear maps $f : U \longrightarrow V$ and $g : V \longrightarrow W$. Then, there exists an exact sequence of vector spaces and linear maps

$$0 \longrightarrow \ker f \xrightarrow{\iota} \ker gf \xrightarrow{f_0} \ker g \xrightarrow{\delta} \operatorname{coker} f \xrightarrow{\overline{g}} \operatorname{coker} gf \xrightarrow{p} \operatorname{coker} f \longrightarrow 0.$$

Here, $\iota$ is the inclusion, $f_0$ is the restriction of $f$, $\delta$ is the restriction of the quotient map $V \longrightarrow \operatorname{coker} f$ to the subspace $\ker g \subseteq V$, $\overline{g}$ is the map induced from $g$ by passage to the quotient and $p$ is that induced by the identity of $W$.

*Proof.* It is easily seen that an application of Proposition 2 to the diagram

$$\begin{array}{ccccccc} & & 0 & \longrightarrow & U & \xrightarrow{1_U} & U & \longrightarrow & 0 \\ & & \downarrow & & f\downarrow & & gf\downarrow & & \\ 0 & \longrightarrow & \ker g & \longrightarrow & V & \xrightarrow{g} & W & & \end{array}$$

provides us with the exact sequence

$$0 \longrightarrow \ker f \xrightarrow{\iota} \ker gf \xrightarrow{f_0} \ker g \xrightarrow{\delta} \operatorname{coker} f \xrightarrow{\overline{g}} \operatorname{coker} gf.$$

Applying once again Proposition 2, this time to the diagram

$$\begin{array}{ccccccccc} & & U & \xrightarrow{f} & V & \longrightarrow & \operatorname{coker} f & \longrightarrow & 0 \\ & & gf\downarrow & & g\downarrow & & \downarrow & & \\ 0 & \longrightarrow & V & \xrightarrow{1_V} & V & \longrightarrow & 0 & & \end{array}$$

is easily seen to give the exact sequence

$$\ker gf \xrightarrow{f_0} \ker g \xrightarrow{\delta} \operatorname{coker} f \xrightarrow{\overline{g}} \operatorname{coker} gf \xrightarrow{p} \operatorname{coker} f \longrightarrow 0.$$

The two exact sequences obtained above may be glued together to yield the exact sequence in the statement of the proposition. □

III. Composition of Fredholm-like linear maps. If $U, V$ are two vector spaces, then we denote by $\mathfrak{F}(U, V)$ the set consisting of those linear maps $f : U \longrightarrow V$ for which both vector spaces $\ker f$ and $\operatorname{coker} f$ are finite dimensional. The index $\operatorname{ind} f$ of such a linear map $f$ is defined by letting $\operatorname{ind} f = \dim\ker f - \dim\operatorname{coker} f \in \mathbb{Z}$.

**Remark 4.** Let $U, V$ be two finite dimensional vector spaces. Then, $\mathfrak{F}(U, V)$ is the set of all linear maps $f : U \longrightarrow V$. Moreover, the index of any linear map $f : U \longrightarrow V$ is equal to the difference $\dim U - \dim V$. This follows from Proposition 1(ii), applied to the exact sequence

$$0 \longrightarrow \ker f \longrightarrow U \xrightarrow{f} V \longrightarrow \operatorname{coker} f \longrightarrow 0.$$

In particular, if $U$ is a finite dimensional vector space, then $\mathfrak{F}(U, U)$ is the set of all linear operators $f$ on $U$ and the index of any such an $f$ is equal to 0.

**Proposition 5.** Let $U, V, W$ be vector spaces and consider two linear maps $f \in \mathfrak{F}(U, V)$ and $g \in \mathfrak{F}(V, W)$. Then, the composition $gf : U \longrightarrow W$ is actually contained in $\mathfrak{F}(U, W)$ and we have an equality $\operatorname{ind} gf = \operatorname{ind} f + \operatorname{ind} g$.

*Proof.* Since the vector spaces $\ker f$ and $\ker g$ are both finite dimensional, the exact sequence of Corollary 3 together with Proposition 1(i) imply that $\ker gf$ is finite dimensional as well. In the same way, the finite dimensionality of $\operatorname{coker} f$ and $\operatorname{coker} g$ implies that $\dim \operatorname{coker} gf < \infty$. Invoking Proposition 1(ii), the exact sequence of Corollary 3 provides us with an equality

$$\dim \ker f - \dim \ker gf + \dim \ker g - \dim \operatorname{coker} f + \dim \operatorname{coker} gf - \dim \operatorname{coker} g = 0.$$

It follows that $\operatorname{ind} f - \operatorname{ind} gf + \operatorname{ind} g = 0$, as needed. $\square$

**Exercise 6.** If $A, B$ are any two abelian groups, then define $\mathbb{F}(A, B)$ to be the set consisting of those additive maps $f : A \longrightarrow B$, for which both groups $\ker f$ and $\operatorname{coker} f$ are finite. We also define the index $v(f)$ of such an additive map $f$ to be the number $\log |\ker f| - \log |\operatorname{coker} f|$. (Here, we denote by $|G|$ the order, i.e. the cardinality, of a finite group $G$.)

Let $A, B, C$ be three abelian groups and consider two additive maps $f \in \mathbb{F}(A, B)$ and $g \in \mathbb{F}(B, C)$. Then, show that the composition $gf : A \longrightarrow C$ is contained in $\mathbb{F}(A, C)$ and we have an equality $v(gf) = v(g) + v(f)$.