Hilbert modules, TRO's and C*-correspondences

(rough notes by A.K.)

1 Hilbert modules and TRO's

1.1 Reminders

Recall[1] the definition of a Hilbert module

Definition 1 *Let A be a C*-algebra. An* **Hilbert C*-module over** *A is a complex vector space E which is a right A-module and is equipped with an A-valued scalar product satisfying*

1. $\langle x, \lambda y + z\rangle = \lambda \langle x, y\rangle + \langle x, y\rangle$
2. $\langle x, y \cdot a\rangle = \langle x, y\rangle a$
3. $\langle x, y\rangle^* = \langle y, x\rangle$
4. $\langle x, x\rangle \in A_+$
5. $\langle x, x\rangle = 0 \Rightarrow x = 0 \qquad (x, y, z \in E,\ a \in A,\ \lambda \in \mathbb{C})$

and which is complete with respect to the norm

$$\|x\|_E := \|\langle x, x\rangle\|_A^{1/2} \qquad (x \in E).$$

An operator $T : E \to F$ between Hilbert C*-modules over A is said to be *adjointable* (written $T \in \mathcal{L}(E, F)$) if there is a linear map $T^* : F \to E$ such that

$$\langle T^* f, e\rangle_E = \langle f, Te\rangle_F \quad \text{for all} \quad e \in E, f \in F.$$

When $E = F$, the space $\mathcal{L}(E) := \mathcal{L}(E, E)$ is a C*-algebra. Recall also the definition of the 'compact' operators: For $x \in F$, $y \in E$ we define the map $\theta_{x,y} : E \to F$ by:

$$\theta_{x,y}(z) = x \langle y, z\rangle .$$

This map has the properties

$$\theta^*_{x,y} = \theta_{y,x}, \quad T\theta_{x,y} = \theta_{Tx,y}, \quad \theta_{x,y}S = \theta_{x,S^*y}, \quad \theta_{x,y}\theta_{u,v} = \theta_{x\langle y,u\rangle,v} = \theta_{x,v\langle u,y\rangle}.$$

The closed linear span of the set $\{\theta_{x,y} : x \in F, y \in E\}$ is a closed subspace of $\mathcal{L}(E, F)$. We call it the space of 'compact' operators and denote it by $\mathcal{K}(E, F)$. When $E = F$ then $\mathcal{K}(E) := \mathcal{K}(E, E)$ is an ideal in the C*-algebra $\mathcal{L}(E)$.

[1] katsnotes, 15 April 2013

1.2 Representing Hilbert modules as TRO's

If E is a Hilbert module over A and $x \in E$ we define two maps

$$L_x : A \to E : a \to xa \qquad D_x : E \to A : y \to \langle x, y \rangle .$$

Then we have

$$(L_x)^* = D_x$$

so that both maps are adjointable (A is a Hilbert module over itself with $\langle x, y \rangle = xy^*$). Clearly, $\|D_x\| = \|x\|$; thus the map $x \to D_x$ is an antilinear isometry into $\mathcal{L}(E, A)$. Moreover, the easily verified identity

$$D_{xa^*} = \theta_{a,x}$$

shows that $\mathcal{K}(E, A)$ is contained in the (closed) range of D. The same identity, together with an approximate unit argument, shows that in fact D maps E *onto* $\mathcal{K}(E, A)$. Taking adjoints, we conclude that

Proposition 1 *The map $x \to L_x : E \to \mathcal{K}(A, E)$ is a linear isometric bijection, and the map $x \to D_x : E \to \mathcal{K}(E, A)$ is an antilinear isometric bijection.*

In particular, $a \to L_a$ is a linear bijection between A and $\mathcal{K}(A)$ and it is easily verified that it is a *-homomorphism. [2]

Thus we may identify E with the linear space $\mathcal{K}(E, A)$ and define its 'adjoint' (or 'conjugate') space $\bar{E}$ to be $\mathcal{K}(A, E)$ (so that $\bar{E}$ is antilinearly isometric to E).

Definition 2 *Given $T \in \mathcal{K}(E)$, $x, y \in E$ and $a \in A$ define the following operator on the Hilbert A-module [3] $E \oplus A$:*

$$\begin{bmatrix} T & x \\ \bar{y} & a \end{bmatrix} : \begin{bmatrix} \xi \\ b \end{bmatrix} \to \begin{bmatrix} T\xi + L_x b \\ D_y(\xi) + L_a b \end{bmatrix} = \begin{bmatrix} T\xi + xb \\ \langle y, \xi \rangle + ab \end{bmatrix}$$

The set of all such operators,

$$\mathbb{L}(E) := \begin{bmatrix} \mathcal{K}(E) & \mathcal{K}(A, E) \\ \mathcal{K}(E, A) & \mathcal{K}(A) \end{bmatrix} \simeq \begin{bmatrix} \mathcal{K}(E) & E \\ \bar{E} & A \end{bmatrix} \subseteq \mathcal{L}(E \oplus A)$$

is a C-subalgebra of $\mathcal{L}(E \oplus A)$, called* the linking algebra of *E.*

Note that $\mathbb{L}(E)$ is in fact a C*-subalgebra of $\mathcal{K}(E \oplus A)$. In fact, $\mathbb{L}(E) = \mathcal{K}(E \oplus A)$. Indeed, it is clear that the four "corners" of $\mathbb{L}(E)$ are bimodules over the appropriate algebras. For instance if $A \in \mathcal{L}(E)$ and $B \in \mathcal{L}(A)$ then for all $\theta_{x,a}$ with $a \in A$ and $x \in E$ we have $A\theta_{x,a}B = \theta_{Ax,B^*a} \in \mathcal{K}(E, A)$. Thus if $p_1, p_2 \in \mathcal{L}(E \oplus A)$ denote the canonical orthogonal projections onto E and A respectively [4] then given $T \in \mathcal{K}(E \oplus A)$ we see for example that $p_1 T p_2|_A \in \mathcal{K}(A, E) \simeq E$, etc.

Thus, if one represents $\mathbb{L}(E)$ faithfully as operators on some Hilbert space H, then E can be identified as a closed linear space of operators on H. Of course E is not in general a subalgebra of $B(H)$, however it is a 'ternary ring of operators' in the following sense

[2] $L_{ab}c = (ab)c = a(bc) = L_a(L_b(c))$ and $(L_a)^*(c) = D_a(c) = \langle a, c \rangle_A = a^*c = L_{a^*}(c)$.

[3] with scalar product $\langle (x, a), (y, b) \rangle = \langle x, y \rangle_E + a^*b$

[4] the easily verified fact that $\langle p_i(x), y \rangle = \langle x, p_i(x) \rangle$ shows that the p_i are adjointable

Definition 3 *A* **ternary ring of operators (TRO)** *is a linear subspace $\mathcal{X}$ of some $B(H)$ (or more generally of a C^*-algebra B) such that*

$$a, b, c \in X \Rightarrow ab^*c \in \mathcal{X}.$$

Indeed, it is immediate that if x, y, z are in E, the corresponding operators X, Y, Z in $\mathbb{L}(E)$ satisfy

$$\begin{aligned} XY^*Z &= \begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix} \left(\begin{bmatrix} 0 & y \\ 0 & 0 \end{bmatrix} \right)^* \begin{bmatrix} 0 & z \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ \bar{y} & 0 \end{bmatrix} \begin{bmatrix} 0 & z \\ 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & \langle y, z \rangle \end{bmatrix} = \begin{bmatrix} 0 & x \langle y, z \rangle \\ 0 & 0 \end{bmatrix} \end{aligned}$$

and $x \langle y, z \rangle$ is in E. Sometimes it is useful to think of the scalar product $\langle y, z \rangle$ as a product of operators, Y^*Z, remembering that this product is not in $E \simeq \mathcal{K}(E, A)$, but in $A \simeq \mathcal{K}(A)$.

2 C*-correspondences

Definition 4 *A* C*-correspondence *is a quadruple (X, B, A, ϕ) (sometimes written ${}_AX_B$ or (X, ϕ)) where*

(a) X is a Hilbert C^*-module over B (so we have maps $X \times X \to B : (\xi, \eta) \to \langle \xi, \eta \rangle$ and $X \times B \to X : (\xi, b) \to \xi b$) but also

*(b) there is a *-homomorphism $\phi = \phi_X : A \to \mathcal{L}_B(X)$.*

The correspondence ${}_AX_B$ is called injective *when ϕ is 1-1 (hence isometric).*

It is called non-degenerate *when $\overline{\text{span}}\{\phi(A)X\} = X$.*

It is called full *when $\overline{\langle X, X \rangle} = B$ (i.e. when $\overline{\text{span}}\{\langle \xi, \eta \rangle : \xi, \eta \in X\} = B$).*

Definition 5 *A* representation *of a C^*-correspondence ${}_AX_B$ into $\mathcal{B}(\mathcal{H})$, or more generally into a C^*-algebra $\mathcal{B}$ is a pair (π, t) where*

$$\begin{aligned} &\pi : A \to \mathcal{B} \quad \textit{is a *-homomoprhism} \\ &t : X \to \mathcal{B} \quad \textit{is a linear map, and} \\ &\pi(a)t(\xi) = t(\phi(a)\xi) \\ &t(\xi)^*t(\eta) = \pi(\langle \xi, \eta \rangle) \qquad a \in A,\ \xi, \eta \in X. \end{aligned}$$

Definition 6 *The* Toeplitz *algebra (or* Toeplitz-Cuntz-Pimsner *algebra) $\mathcal{T}(X)$ of a C^*- correspondence ${}_AX_A$ is defined to be the C^*-algebra $C^*(\bar{\pi}, \bar{t})$ generated by the* universal representation $(\bar{\pi}, \bar{t})$*: it has the universal property that whenever (π, t) is a representation of X, there is a *-epimorphism $\rho : \mathcal{T}(X) \to C^*(\pi, t)$ satisfying $\pi = \rho \circ \bar{\pi}$ and $t = \rho \circ \bar{t}$.*

The Tensor algebra *$\mathcal{T}^+(X)$ is the norm-closed (non-selfadjoint) subalgebra of $\mathcal{T}(X)$ generated by $\{\bar{\pi}(a) : a \in A\}$ (a selfadjoint subalgebra) together with $\{\bar{t}(\xi) : \xi \in X\}$ (a non-selfadjoint subspace)*

The Toeplitz algebra can be defined as the direct sum of 'sufficiently many' representations (π, t) of X. But there is an explicit representation, which can be shown to possess the required universal property, and to be unique (up to *-isomorphism) with that property. This is the so called 'Fock representation'.

To construct it, we need the notion of internal tensor product.

2.1 The internal tensor product

Let E_A be a Hilbert C*-module over A and let ${}_AF_B$ be a C*-correspondence with $\phi : A \to \mathcal{L}_B(F)$. We construct the internal tensor product $E \otimes_\phi F$ in three stages:

(i) Let $E \odot F$ be the algebraic tensor product, and define the B-valued sesquilinear form $\langle \cdot, \cdot \rangle$ on $E \odot F$ as follows

$$\langle x \otimes y, u \otimes v \rangle := \langle y, \phi(\langle x, u \rangle_E) v \rangle_F \ ,$$

$$\text{i.e.} \qquad \left\langle \sum_i x_i \otimes y_i, \sum_j u_j \otimes v_j \right\rangle := \sum_{i,j} \left\langle y_i, \phi(\langle x_i, u_j \rangle_E) v_j \right\rangle_F \ .$$

This is well defined. [5] In fact $\langle \cdot, \cdot \rangle$ is positive semidefinite. To see this observe that it may be written

$$\left\langle \sum_i x_i \otimes y_i, \sum_j u_j \otimes v_j \right\rangle := \left\langle \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}, [\phi(\langle x_i, u_j \rangle_E)]_{M_n(A)} \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} v_j \right\rangle_F \ .$$

and so, if $[a_{i,j}]$ is in $M_n(A)_+$, then $[\phi(a_{i,j})]$ is positive because ϕ, being a *-homomorphism, is completely positive. [6]

(ii) The quotient $E \otimes_A F = (E \odot F)/N$ is *balanced* over A: thus in $E \otimes_A F$ we have

$$ua \otimes v = u \otimes \phi(a)v.$$

Indeed, $\langle x \otimes y, ua \otimes v \rangle = \langle y, \phi(\langle x, ua \rangle_E) v \rangle_F = \langle y, \phi(\langle x, u \rangle_E\, a) v \rangle_F = \langle y, \phi(\langle x, u \rangle_E) \phi(a) v \rangle_F = \langle x \otimes y, u \otimes \phi(a) v \rangle_F$ so $\langle x \otimes y, ua \otimes v - u \otimes \phi(a) v \rangle = 0$ for all $x \otimes y$.

Moreover $E \otimes_A F$ becomes a right B-module by defining $(x \otimes y) \cdot b = x \otimes (yb)$ and $\langle \cdot, \cdot \rangle$ induces a scalar product on $E \otimes_A F$ satisfying $\langle z, w \cdot b \rangle = \langle z, w \rangle b$ for $z, w \in E \otimes_A F$ and $b \in B$ (recall that the scalar product on $E \otimes_A F$ is B-valued).

(iii) The completion of $E \otimes_A F$ with respect to the norm $\|z\| := \|\langle z, z \rangle\|_B^{1/2} \qquad (z \in E \otimes_A F)$ is a C*-Hilbert module over B (the right action of B extends by continuity). This is called *the internal tensor product of E_A and ${}_AF_B$* and we will denote it by $E \otimes_\phi F$ or $E \otimes F$.

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[5] For instance if $\sum_j u_j \otimes v_j = 0$ then for all $x \in E$, since $(u, v) \to \phi(\langle x, u \rangle_E) v$ is bilinear on $E \times F$ we have $\sum_j \phi(\langle x, u_j \rangle_E) v_j = 0$ and so $\left\langle x \otimes y, \sum_j u_j \otimes v_j \right\rangle = 0$ for all $x \otimes y$.

[6] in fact whenever ϕ is just a completely positive map the above formula defines a semi-inner product on $E \odot F$

For $\xi \in E$, define

$$T_\xi : F \to E \otimes_\phi F : \eta \to \xi \otimes \eta$$

One can check that

$$T_\xi^* : E \otimes_\phi F \to F : x \otimes y \to \phi(\langle \xi, x \rangle_E) y \, .$$

Indeed,

$$\begin{aligned}\left\langle T_\xi^*(x \otimes y), \eta \right\rangle_F &= \langle x \otimes y, T_\xi(\eta) \rangle_{E \otimes F} = \langle x \otimes y, \xi \otimes \eta \rangle_{E \otimes F} \\ &= \langle y, \phi(\langle x, \xi \rangle) \eta \rangle_F = \langle \phi(\langle x, \xi \rangle)^* y, \eta \rangle_F \\ &= \left\langle \phi(\langle x, \xi \rangle^*) y, \eta \right\rangle_F = \langle \phi(\langle \xi, x \rangle) y, \eta \rangle_F \, .\end{aligned}$$

It follows that

$$T_\xi T_\eta^* : x \otimes y \to \phi(\langle \eta, x \rangle) y \to \xi \otimes \phi(\langle \eta, x \rangle) y = \theta_{\xi,\eta}(x) \otimes y$$

because, if we write a for $\langle \eta, x \rangle$,

$$\xi \otimes \phi(\langle \eta, x \rangle) y = \xi \otimes \phi(a) y = \xi a \otimes y = \xi \langle \eta, x \rangle \otimes y = \theta_{\xi,\eta}(x) \otimes y.$$

Thus

$$T_\xi T_\eta^* = \theta_{\xi,\eta} \otimes I \in \mathcal{L}(E \otimes_\phi F).$$

Also,

$$\begin{aligned} & T_\eta^* T_\xi : x \to \xi \otimes x \to \phi(\langle \eta, \xi \rangle) x \\ \text{so} \quad & T_\eta^* T_\xi = \phi(\langle \eta, \xi \rangle) \in \mathcal{L}(E). \end{aligned}$$

Definition 7 *If ${}_A E_A$ and ${}_A F_A$ are both C^*-correspondences via ϕ, then $E \otimes_\phi F$ becomes a C^*-correspondence over A via $\tilde{\phi}$ defined by*

$$\tilde{\phi}(a) = \phi(a) \otimes I, \qquad a \in A.$$

2.2 The Fock representation

Let ${}_A E_A$ be a C*-correspondence and define a sequence of C*-correspondences over A as follows

$$E^{\otimes 1} = E, \quad E^{\otimes n+1} = E \otimes_{\phi_n} E^{\otimes n}.$$

Thus each $E^{\otimes n}$ is a C*-correspondence over A with

$$\begin{aligned} \langle \xi_1 \otimes \xi_2 \otimes \ldots \otimes \xi_n, \eta_1 \otimes \eta_2 \otimes \ldots \otimes \eta_n \rangle &= \langle \xi_2 \otimes \ldots \otimes \xi_n, \phi(\langle \xi_1, \eta_1 \rangle)(\eta_2 \otimes \ldots \otimes \eta_n) \rangle \\ (\xi_1 \otimes \xi_2 \otimes \ldots \otimes \xi_n) \cdot a &= \xi_1 \otimes \xi_2 \otimes \ldots \otimes (\xi_n a) \\ \text{and} \quad \phi_n(a)(\xi_1 \otimes \xi_2 \otimes \ldots \otimes \xi_n) &= (\phi(a)\xi_1) \otimes \xi_2 \otimes \ldots \otimes \xi_n \qquad \text{i.e.} \quad \phi_n = \tilde{\phi} \end{aligned}$$

Definition 8 (The Fock space $\mathcal{F}(E)$) *This is defined to be the direct sum $\bigoplus_k E^{\otimes k}$ of the sequence of Hilbert C*-modules over A where $E^{\otimes 0} := A$. Thus*

$$\begin{aligned}\mathcal{F}(E) &= A \oplus E \oplus (E \otimes_\phi E) \oplus \dots \\ &:= \{x = (x(k)) \in \prod_k E^{\otimes k} : \sum_k \langle x(k), x(k)\rangle_{E^{\otimes k}} \text{ converges in the norm of } A\}.\end{aligned}$$

For $\xi \in E$, recall the map $T_\xi : F \to E \otimes_\phi F : \eta \to \xi \otimes \eta$. When $F = E^{\otimes n}$, we denote this by $t_n(\xi)$. Thus

$$t_n(\xi)(\xi_1 \otimes \dots \otimes \xi_n) = \xi \otimes \xi_1 \otimes \dots \otimes \xi_n \in E^{\otimes n+1}.$$

Note that since $t_n(\xi)^* t_n(\xi) = \phi_n(\langle \xi, \xi\rangle)$, we have $\|t_n(\xi)\|^2 = \|\phi(\langle \xi, \xi\rangle)\| \leqslant \|\xi\|^2$. We define the *creation operator* on $\mathcal{F}(E)$ by

$$t_\infty(\xi)(a, x_1, x_2 \dots) = (0, \xi a, \xi \otimes x_1, \xi \otimes x_2, \dots)$$

(here $a \in A$ and $x_k \in E^{\otimes k}$) and we observe that the map

$$t_\infty : E \to \mathcal{L}(\mathcal{F}(E))$$

is linear and contractive. We also define the *-homomorphism

$$\phi_\infty : A \to \mathcal{L}(\mathcal{F}(E)) : a \to \operatorname{diag}(a, \phi(a), \phi_2(a), \dots).$$

We verify the conditions for the pair (ϕ_∞, t_∞) to be a representation of the C*-correspondence E:

$$\begin{aligned} t_\infty^*(\eta) t_\infty(\xi) &= \phi_\infty(\langle \eta, \xi\rangle) \\ \phi_\infty(a) t_\infty(\xi) &= t_\infty(\phi(a)\xi). \end{aligned}$$

The first relation is immediate from $t_n^*(\eta) t_n(\xi) = \phi_n(\langle \eta, \xi\rangle)$. We verify the second:

$$\begin{aligned}\phi_\infty(a) t_\infty(\xi) : (b, x_1, x_2, \dots) &\to (0, \xi b, \xi \otimes x_1, \xi \otimes x_2, \dots) \to (0, \phi(a)\xi b, \phi(a)\xi \otimes x_1, \phi_2(a)(\xi \otimes x_2), \dots) \\ &= (0, \phi(a)\xi b, \phi(a)\xi \otimes x_1, \phi(a)\xi \otimes x_2, \dots) = (0, (\phi(a)\xi) b, (\phi(a)\xi) \otimes x_1, (\phi(a)\xi) \otimes x_2, \dots) \\ &= t_\infty(\phi(a)\xi)(b, x_1, x_2, \dots)\end{aligned}$$

Thus the pair (ϕ_∞, t_∞) is a representation of the C*-correspondence E. Moreover, it is injective, since ϕ_∞ is 1-1. This is obvious since $\phi_\infty(a) = \operatorname{diag}(a, \phi(a), \phi_2(a), \dots)$.

Theorem 2 *The C*-algebra generated by the Fock representation (ϕ_∞, t_∞) is *-isomorphic to the Toeplitz algebra $\mathcal{T}(X)$.*

2.3 The Cuntz-Pimsner algebra

Definition 9 (The Katsura ideal) *For a C^*-correspondence ${}_AE_A$, we define an ideal of A by*

$$J_E = \{a \in A : \phi(a) \in \mathcal{K}(E) \ \ \textit{and} \ \ ab = 0 \ \forall b \in \ker\phi\}$$

Note that since $\mathcal{F}(E)$ is a right A-module, it is also a right J_E-module, i.e. $\mathcal{F}(E)J_E \subseteq \mathcal{F}(E)$. Consider the ideal of $\mathcal{L}(\mathcal{F}(E))$

$$\mathcal{K}(\mathcal{F}(E)J_E) = \overline{\text{span}}\{\theta_{xa,y} \in \mathcal{K}(\mathcal{F}(E)) : x, y \in \mathcal{F}(E), a \in J_E\}.$$

Definition 10 (The Cuntz-Pimsner algebra) *This is the quotient*

$$\mathcal{O}(E) = \mathcal{T}(E)/\mathcal{K}(\mathcal{F}(E)J_E).$$

This also has a universal property ...

(... to be continued)