# HILBERT C*-MODULES
# SECOND LECTURE
# OPERATORS ON HILBERT C*-MODULES

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## 1. Adjointable Operators

**Definition 1.** *Let $A$ be a $C^*$-algebra and $E$ a Hilbert $C^*$-module over $A$. A map $T : E \to E$ is called adjointable if there exists a map $T^* : E \to E$ such that $\langle Tx, y\rangle = \langle x, T^*y\rangle$ for all $x, y$ in $E$.*

**Remark 1.** *If follows from the definition that if $T$ is adjointable, $T^*$ is adjointable and $\langle T^*x, y\rangle = \langle x, Ty\rangle$. That is $(T^*)^* = T$.*

**Proposition 2.** *Let $T$ be an adjointable map. Then*

(1) *$T$ is a linear module map.*
(2) *$T$ is bounded.*

*Proof.* (1) Linearity:
If $x, y, z \in E$ and $\lambda, \mu \in \mathbb{C}$ we have:

$$\langle T(\lambda x + \mu y), z\rangle = \langle \lambda x + \mu y, T^*z\rangle = \overline{\lambda}\,\langle x, T^*z\rangle + \overline{\mu}\,\langle y, T^*z\rangle =$$
$$\overline{\lambda}\,\langle Tx, z\rangle + \overline{\mu}\,\langle Ty, z\rangle = \langle \lambda T(x) + \mu T(y), z\rangle$$

and so $T(\lambda x + \mu y) = \lambda T(x) + \mu T(y)$.
$T$ is a module map:
If $x, y \in E$ and $a \in A$ we have:

$$\langle T(xa), y\rangle = \langle xa, T^*y\rangle = a^*\,\langle x, T^*y\rangle = a^*\,\langle T(x), y\rangle =$$
$$\langle T(x)a, y\rangle$$

and so $T(xa) = T(x)a$.

(2) $T$ is bounded: Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in $E$. Assume there exist $x, z \in E$ such that $x_n \to x$ and $Tx_n \to z$. Let $y \in E$. We have:

$$\langle T(x_n), y\rangle \to \langle z, y\rangle$$

and also

$$\langle Tx_n, y\rangle = \langle x_n, T^*y\rangle \to$$
$$\langle x, T^*y\rangle = \langle Tx, y\rangle$$

and so $T(x) = z$. Hence $T$ is bounded by the Closed Graph Theorem.

□

**Proposition 3.** *Let $T$ and $S$ be adjointable operators and $\lambda \in \mathbb{C}$. Then*

(1) $(T + S)^* = T^* + S^*$.

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(2) $(\lambda T)^* = \overline{\lambda} T$.
(3) $TS$ *is adjointable and* $(TS)^* = S^* T^*$.

We denote by $\mathcal{L}(E)$ the algebra of adjointable operators on $E$.

**Proposition 4.** $\mathcal{L}(E)$ *is a* $C^*$*-algebra.*

*Proof.* Let $T \in \mathcal{L}(E)$. We have

$$\|T^*T\| \leq \|T^*\|\|T\|$$

and also

$$\|T^*T\| \geq \sup_{x\in E, \|x\|\leq 1} \{\langle T^*Tx, x\rangle\} = \sup_{x\in E, \|x\|\leq 1} \{\langle Tx, Tx\rangle\} = \|T\|^2. \tag{1}$$

It follows that

$$\|T\| \leq \|T^*\|$$

and since $T^{**} = T$ we obtain

$$\|T\| \leq \|T^*\| \leq \|T\| \Rightarrow \|T^*\| = \|T\|.$$

By the inequality 1 above we then have:

$$\|T\|^2 \leq \|T^*T\| \leq \|T^*\|\|T\| = \|T\|^2,$$

and finally

$$\|T\|^2 = \|T^*T\|.$$

We show that $\mathcal{L}(E)$ is complete. Let $\{T_n\}_{n\in\mathbb{N}}$ be a Cauchy sequence in $\mathcal{L}(E)$. Since the space of bounded linear operators on $E$ is a Banach space, $\{T_n\}_{n\in\mathbb{N}}$ converges to a linear operator $T$ and $\{T_n^*\}_{n\in\mathbb{N}}$ converges to a linear operator $\overline{T}$. We show that $T$ is adjointable and $T^* = \overline{T}$. We have for $y \in E$:

$$\langle Tx, y\rangle = \lim \langle T_n x, y\rangle = \lim \langle x, T_n^* y\rangle = \left\langle x, \overline{T} y\right\rangle.$$

So, $T^* = \overline{T}$ and $\mathcal{L}(E)$ is complete. □

**Example 5.** *Let* $A = C([0,1])$, $J = \{f \in A : f(0) = 0\}$. *Consider the Hilbert* $C^*$*-module* $A \oplus E$ *over* $A$. *Define* $T$ *as follows:* $T(f,g) = (g, 0)$. *We show that* $T$ *is not adjointable. Assume there exists a map* $S : E \to E$ *such that* $\langle Tx, y\rangle = \langle x, Sy\rangle$ *for all* $x, y$ *in* $E$. *Then* $S(1,0) = (f,g)$ *for some* $f \in A$, $g \in J$. *Hence for all* $(h,k) \in E$ *we have*

$$\langle T(h,k), (1,0)\rangle = \langle (h,k), S(1,0)\rangle \Rightarrow \overline{k} = \overline{h} f + \overline{k} g.$$

*Set* $f = 0$. *This forces* $\overline{k} = \overline{k} g$ *for every* $k \in J$. *Hence* $g = 1$. *Since* $g(0) = 0$ *we obtain a contradiction.*

**Proposition 6.** *Let* $T$ *in* $\mathcal{L}(E)$. *Then for all* $x \in E$,

$$\langle Tx, Tx\rangle \leq \|T\|^2 \langle x, x\rangle.$$

*Proof.* The operator $\|T\|^2 I - T^*T$ is positive in the $C^*$-algebra $\mathcal{L}(E)$ and hence there exists $S \in \mathcal{L}(E)$ such that $\|T\|^2 I - T^*T = S^*S$. We have for $x \in E$:

$$\langle Sx, Sx\rangle \geq 0 \Rightarrow \langle S^*Sx, x\rangle \geq 0 \Rightarrow \left\langle (\|T\|^2 I - T^*T)x, x\right\rangle \geq 0$$

$$\|T\|^2 \langle x, x\rangle - \langle Tx, Tx\rangle \geq 0.$$

□

## 2. Compact operators on Hilbert modules

**Definition 2.** *Let $A$ be a $C^*$-algebra and $E$ a Hilbert $C^*$-module over $A$. Let $x, y$ in $E$. Define the map $\Theta_{x,y} : E \to E$ by:*

$$\Theta_{x,y}(z) = x \langle y, z \rangle .$$

**Proposition 7.** *Let $A$ be a $C^*$-algebra and $E$ a Hilbert $C^*$-module over $A$. Then for every $x, y$ in $E$ the map $\Theta_{x,y} : E \to E$ is adjointable and*

$$\Theta^*_{x,y} = \Theta_{y,x}.$$

*Proof.* For $z, w \in E$ we have:

$$\langle \Theta_{x,y} z, w \rangle = \langle x \langle y, z \rangle , w \rangle = \langle y, z \rangle^* \langle x, w \rangle =$$
$$\langle z, y \rangle \langle x, w \rangle = \langle z, y \langle x, w \rangle \rangle = \langle z, \Theta_{y,x} w \rangle$$

□

**Proposition 8.** *Let $A$ be a $C^*$-algebra and $E$ a Hilbert $C^*$-module over $A$. The closed linear span of the set $\{\Theta_{x,y} : x \in E, y \in E\}$ is a closed ideal in $\mathcal{L}(E)$. We call it the algebra of compact operators on $E$ and denote it by $\mathcal{K}(E)$.*

*Proof.* Let $T \in \mathcal{L}(E)$ and $x, y \in E$. We have:

$$T\Theta_{x,y} = \Theta_{Tx,y}$$

and

$$\Theta_{x,y} T = \Theta_{x,T^*y}.$$

The proposition follows. □

**Example 9.** *Let $H$ be a Hilbert space, $A = \mathbb{C}$ and consider the Hilbert space $H$ as a Hilbert $C^*$-module over $A$. Then the algebra of adjointable operators on the Hilbert $C^*$-module $H$ over $A$ is the algebra of bounded linear operators on $H$, and the algebra of compact operators on the Hilbert $C^*$-module $H$ over $A$ is the algebra of compact operators on the Hilbert space $H$.*

**Example 10.** *Let $A$ be a $C^*$-algebra and consider the Hilbert $C^*$-module $A$ over $A$. Consider the map $L_a : A \to A$ defined by $L_A(x) = ax$. Then $L_a$ is adjointable with adjoint $L_{a^*}$ and $||L_a|| = 1$. Thus the map $a \to L_a$ is an isometric homomorphism from $A$ onto a closed $C^*$-subalgebra $\mathrm{Im}L$ of $\mathcal{L}(E)$. Since $\Theta_{a,b} = L_{ab^*}$, $\mathrm{Im}L$ contains $\mathcal{K}(A)$. On the other hand, if $a \in A$ and $\{u_i\}_{i \in I}$ is a contractive approximate identity for $A$, we have $L_{u_i a} \to L_a$ and since $L_{u_i a}$ is in $\mathcal{K}(A)$ we see that $L_a$ is in $\mathcal{K}(A)$. Thus $\mathrm{Im}L$ is contained in $\mathcal{K}(A)$. We conclude that $\mathcal{K}(A) = \mathrm{Im}L$ and so $\mathcal{K}(A)$ is isomorphic to $A$.*

**Example 11.** *Let $A$ be a unital $C^*$-algebra and consider the Hilbert $C^*$-module $A$ over $A$. Let $T$ be an adjointable operator on $A$. Then $T(a) = T(1a) = T(1)a$ and $T = L_{T(1)}$. The map $a \to L_a$ is an isomorphism from $A$ onto $\mathcal{L}(E)$. Hence we have $\mathcal{L}(E) = \mathcal{K}(E) \simeq A$.*

## References


[1] Iain Raeburn and Dana P. Williams. *Morita equivalence and continuous-trace $C^*$-algebras*, volume 60 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 1998.

[2] N. E. Wegge-Olsen. *$K$-theory and $C^*$-algebras.* Oxford Science Publications. The Clarendon Press Oxford University Press, New York, 1993. A friendly approach.