HILBERT C*-MODULES
3RD, 4TH AND 5TH LECTURES
MULTIPLIER ALGEBRAS

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1. Unitization

Definition 1. *Let A be a C^*-algebra and I an ideal of A. The ideal I is essential if $I \cap J \neq \{0\}$ for every ideal J of A, $J \neq \{0\}$.*

Proposition 1. *The following are equivalent for an ideal I of A.*

(1) *I is essential.*
(2) *If $a \in A$ and $aI = \{0\}$ then $a = 0$.*

Example 2. *Let X be a locally compact Hausdorff space. Consider the C^*-algebra $C_0(X)$. If I is an ideal of $C_0(X)$ there exists an open set U_I such that $I = \{f \in C_0(X) : f(x) = 0, x \notin U_I\}$. An ideal I is essential if and only if U_I is dense in X.*

Definition 2. *A unitization of a C^*-algebra A is a unital C^*-algebra B and an injective homomorphism $i : A \to B$ such that $i(A)$ is an essential ideal in B.*

Remark 3. *If A is unital and B is a unitization of A, then $A = B$.*

Proof. Let 1 be the unit of A and b the unit of B. If $a \in A$ we have $(b-1)a = ba - 1a = 0$ and hence $(b-1)A = \{0\}$. By Proposition 1 $b = 1$ and so $A = B$. $\square$

Example 4. *Let A be a C^*-algebra. Set $A^1 = A \oplus \mathbb{C}$. Define $(a,\lambda)(b,\mu) = (ab + \mu a + \lambda b, \lambda\mu)$ and $(a,\lambda)^* = (a^*, \overline{\lambda})$. Consider the embedding $L : A \to \mathcal{K}(A)$. (L_a is the operator defined by $L_a(x) = ax$ for x in A). Define $\tilde{L} : A^1 \to \mathcal{L}(A)$ by $\tilde{L}((a,\lambda)) = L_a + \lambda I$. Then, the image of A^1 by $\tilde{L}$ is closed in $\mathcal{L}(A)$ and so it is a C^*-algebra. Define the norm on A^1 by $\|(a,\lambda)\| = \|L_a + \lambda I\|$. Then, A^1 with this norm is a C^*-algebra and is a unitization of A.*

Example 5. *Let H be a Hilbert space and $\mathcal{K}(H)$ the algebra of compact operators on H. Then the subalgebra $\mathcal{K}(H) + \mathbb{C}I$ of $\mathcal{B}(H)$ is closed in $\mathcal{B}(H)$ and is a unitization of $\mathcal{K}(H)$.*

Example 6. *Let A be a C^*-algebra and consider the Hilbert C^*-module A over A. Consider the map $L : A \to \mathcal{K}(A)$. Then $\mathcal{L}(A)$ is a unitization of A. One has to show that $\mathcal{K}(A)$ is an essential ideal of $\mathcal{L}(A)$. Let $T \in \mathcal{L}(A)$ and assume that $T\Theta_{x,y} = 0$ for every $x, y \in A$. Then $\Theta_{Tx,y} = 0$ for every $x, y \in A$ which implies that $Tx = 0$ for every $x \in A$ and so $T = 0$. From Proposition 1 we see that $\mathcal{K}(A)$ is an essential ideal of $\mathcal{L}(A)$.*

Definition 3. *Let X be a locally compact Hausdorff space. A compactification of X is a compact Hausdorff space Y and a map $i : X \to Y$ such that X is homeomorphic to $i(X)$.*

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Example 7. *If X is a locally compact Hausdorff space the one point compactification of X is a compactification. The Stone-Cěch compactification of X is also a compactification.*

Let X be a locally compact Hausdorff space and Y a compactification of X. If $i : X \to Y$ is the embedding of X into Y, define: $i_* : C_0(X) \to C(Y)$ by

$$i_*f(y) = \begin{cases} 0 & \text{if } y \notin i(X) \\ f(x) & \text{if } y = i(x) \in i(X) \end{cases}$$

Then, $C(Y)$ and i_* is a unitization of $C_0(X)$. If Y is the one-point compactification of X, then $C_0(X)^1 = C(Y)$.

2. MULTIPLIER ALGEBRAS

Definition 4. *A unitization (B, i) of a C^*-algebra A is maximal if whenever C is a C^*-algebra, $j : A \to C$ a homomorphism such that $j(A)$ is an essential ideal of C, then there exists an homomorphism $\phi : C \to B$ such that $\phi j = i$.*

It is not obvious from the definition that a maximal unitization of a C^*-algebra exists. This is proved in the following:

Theorem 8. *Let A be a C^*-algebra. The C^*-algebra $(\mathcal{L}(A), i)$ (where $i(a) = L_a$) is a maximal unitization of A. Moreover it (B, j) is another maximal unitization, there exists an isomorphism $\phi : B \to \mathcal{L}(A)$ such that $\phi j = i$.*

Definition 5. *We will refer to $\mathcal{L}(A)$ as the multiplier algebra of A and denote it by $M(A)$.*

Definition 6. *Let A be a C^*-algebra and E a Hilbert C^*-module over A. Let B be a C^*-algebra and α an homomorphism $\alpha : B \to \mathcal{L}(E)$ We say that α is a non-degenerate homomorphism if $\alpha(B)E$ is dense in E.*

Proposition 9. *Let A be a C^*-algebra and E a Hilbert C^*-module over A. Let C be a C^*-algebra and B an ideal of C. Denote by $i : B \to C$ the canonical embedding. If $\alpha : B \to \mathcal{L}(E)$ is a non-degenerate homomorphism, then there is a unique homomorphism $\overline{\alpha} : C \to \mathcal{L}(E)$ which extends α, i.e. $\overline{\alpha} i = \alpha$. If B is an essential ideal in C and α is injective, then $\overline{\alpha}$ is injective.*

Proof. Let $n \in \mathbb{N}$, $b_i \in B$, $x_i \in E$ for $i = 1, 2, .., n$ and $c \in C$. Consider the map $\sum_{i=1}^n \alpha(b_i)x_i \to \sum_{i=1}^n \alpha(cb_i)x_i$. Take an approximate unit e_λ of B. We have

$$\|\sum_{i=1}^n \alpha(cb_i)x_i\| = \lim \|\sum_{i=1}^n \alpha(ce_\lambda b_i)x_i\| \leq$$

$$\limsup \|\alpha(ce_\lambda)\|\|\sum_{i=1}^n \alpha(b_i)x_i\| \leq \limsup \|ce_\lambda\|\|\sum_{i=1}^n \alpha(b_i)x_i\| \leq \|c\|\|\sum_{i=1}^n \alpha(b_i)x_i\|.$$

So this map is bounded on $\alpha(B)E$ and extends to a bounded operator $\overline{\alpha}(c)$ on E. This map is adjointable with adjoint $\overline{\alpha}(c^*)$. Clearly $\overline{\alpha}$ is a homomorphism from C into $\mathcal{L}(E)$ and is unique since $\alpha(B)E$ is dense in E. (If β is another such map, it must agree with $\overline{\alpha}$ on $\alpha(B)E$ and hence on E). Assume now that α is injective and B is essential in C. Then we have $\mathrm{Ker}\alpha \cap B = \{0\}$ since B is injective. But $\mathrm{Ker}\overline{\alpha} \cap B = \mathrm{Ker}\alpha \cap B$ and so $\mathrm{Ker}\overline{\alpha} \cap B = \{0\}$. This implies $\mathrm{Ker}\overline{\alpha} = \{0\}$ since B is essential in C. $\square$

Lemma 10. *Let A and B be C^*-algebras and $\phi : B \to M(A)$ be a non-degenerate homomorphism. Then, there exists a unique homomorphism $\overline{\phi} : M(B) \to M(A)$ which extends ϕ, i.e. such that $\overline{\phi}(b) = \phi(b)$.*

Proof. Apply Proposition 9.

□

Lemma 11. *Let A be a C^*-algebra and (B, i) a maximal unitization of A. Let C be a C^*-algebra and $j : A \to C$ an injective homomorphism such that $j(A)$ is essential in C. Then there exists a unique homomorphism $\phi : C \to B$ such that $\phi j = i$. Moreover ϕ is injective.*

Proof. There exists an homomorphism $\phi : C \to B$ such that $\phi j = i$, since (B, i) a maximal unitization of A. Since $j(A)$ is essential, ϕ is injective since $\mathrm{Ker}\phi \cap j(A) = \{0\}$. Hence, ϕ is injective. If ψ is another homomorphism ψ such that $\psi j = i$ and $c \in C$, $a \in A$ we have:

$$\begin{gathered}(\phi(c) - \psi(c))i(a) = \phi(c)i(a) - \psi(c)i(a) = \\ \phi(c)\phi(j(a)) - \psi(c)\psi(j(a)) = \\ \phi(cj(a)) - \psi(cj(a)) = 0,\end{gathered}$$

since $cj(a) \in j(A)$ and $\phi = \psi$ on $j(A)$. We conclude that $(\phi(c) - \psi(c))j(A) = \{0\}$ and since $j(A)$ is essential this implies that $\phi(c) = \psi(c)$.

□

Proof of Theorem

We show first the uniqueness. Let (B, j) be a maximal unitization of A. Then by the maximality of $\mathcal{L}(A)$ there exists an homomorphism $\phi : B \to \mathcal{L}(A)$ such that $\phi j = i$. By the maximality of B there exists an homomorphism $\psi : \mathcal{L}(A) \to B$ such that $\psi i = j$. Now, $\phi\psi : \mathcal{L}(A) \to \mathcal{L}(A))$ is an homomorphism and satisfies $\phi\psi i = \phi j = i$. But we also have $i_{\mathcal{L}(A)} i = i$ where $i_{\mathcal{L}(A)}$ is the identity map on $\mathcal{L}(A)$. By Lemma 11 we have that $i_{\mathcal{L}(A)} = \phi\psi$. In the same way we prove that $\psi\phi = i_B$ and hence ϕ is an isomorphism.

We show that $\mathcal{L}(A)$ is a maximal unitization. Let C be a C^*-algebra, $j : A \to C$ a homomorphism such that $j(A)$ is an essential ideal of C. Since the map $i : A \to \mathcal{L}(A)$ is a non-degenerate homomorphism, Proposition 9 implies that there exists a map ϕ such that $\phi j = i$. Since i is injective, we conclude that ϕ is injective.

□

The following proposition provides concrete realizations of the multiplier algebra of a C^*-algebra A as an algebra of multipliers.

Proposition 12. *Let A and C be C^*-algebras and E a Hilbert C^*-module over C. Assume that $\alpha : A \to \mathcal{L}(E)$ is an injective, non-degenerate homomorphism. Then α extends to an isomorphism α of $M(A)$ onto $B = \{T \in \mathcal{L}(E) : T\alpha(A) \subseteq \alpha(A),\ \alpha(A)T \subseteq \alpha(A)\}$.*

Proof. It is clear that $\alpha(A)$ is an ideal of B. We show that it is essential. Let $T \in B$ such that $T\alpha(A) = \{0\}$. Then $T\alpha(A)E = \{0\}$ and since α is non-degenerate, $T(E) = \{0\}$ and so $T = 0$. To prove the proposition we only have to show that $\alpha : A \to B$ is a maximal unitization. Assume that there exists a C^*-algebra D and an injective homomorphism $j : A \to D$ such that $j(A)$ is an essential ideal of D. By Proposition 9 there exists an injective homomorphism $\overline{\alpha} : D \to \mathcal{L}(E)$ such that $\overline{\alpha} j = \alpha$. We show that $\overline{\alpha}(D) \subseteq B$. If $d \in D$ and $a \in A$ we have

$$\overline{\alpha}(d)\alpha(a) = \overline{\alpha}(d)\overline{\alpha}(j(a)) = \overline{\alpha}(dj(a)).$$

Since $dj(a) \in j(A)$, $\overline{\alpha}(dj(a)) \in \alpha(A)$ and hence $\overline{\alpha}(d)\alpha(a) \in \alpha(A)$. Similarly we see that $\alpha(a)\overline{\alpha}(d) \in \alpha(A)$ and so $\overline{\alpha}(d) \in B$. □

Corollary 13. *Let A be a C^*-algebra. Then $M(\mathcal{K}(A)) = \mathcal{L}(A)$.*

Proof. What we need to show is that the embedding $i : \mathcal{K}(E) \to \mathcal{L}(E)$ is non-degenerate. Let T_λ be an approximate unit for $\mathcal{K}(E)$. It follows from Theorem 15 that it suffices to show that $T_\lambda x \to x$ for every $x \in E$ of the form $y \langle y, y\rangle$. We have

$$T_\lambda x = T_\lambda(y \langle y, y\rangle) = T_\lambda \Theta_{y,y}(y) =$$
$$(T_\lambda \Theta_{y,y})(y) \to \Theta_{y,y}(y) = x.$$

□

Proposition 14. (1) *Let H be a Hilbert space. Then $M(\mathcal{K}(H)) = \mathcal{B}(H)$.*

(2) *Let T be a locally compact Hausdorff space. Then $M(C_0(T)) = C_b(T) = C(\beta T)$ where $C_b(T)$ is the space of bounded continuous functions on T and βT is the Stone Cěch compactification of T.*

Proof. (1) Follows from Corrolary 13.

(2) It is easy to see that the ideal $C_0(T)$ is an essential ideal in $C_b(T)$. We are going to prove that each multiplier is given by a function in $C_b(T)$. We consider the representation μ of $C_0(T)$ and $C_b(T)$ as multiplication operators on $\ell^2(T)$, so that $(\mu(f)h)(t) = f(t)h(t)$. This representation of $C_0(T)$ is injective and non-degenerate. Hence, it is enough to prove that if $m \in \mathcal{B}(\ell^2(T))$ is such that $m\mu(f) \in \mu(C_0(T))$ and $\mu(f)m \in \mu(C_0(T))$ for all $f \in C_0(T)$, then $m \in \mu(C_b(T))$. So we may assume that for each $f \in C_0(T)$ there exists a function $mf \in C_0(T)$ such that $m\mu(f) = \mu(mf)$. Let $t \in T$. We show that if $f \in C_0(T)$, $g \in C_0(T)$ satisfy $f(t) = g(t) = 1$ then $mf(t) = mg(t)$. Let $h \in \ell^2(T)$ be the function defined by

$$h(s) = \begin{cases} 1 & \text{if } s = t \\ & \text{if } s \neq t \end{cases}$$

Then

$$|mf(t) - mg(t)| = |mf(t)h(t) - mg(t)h(t)| = |(\mu(mf)h)(t) - (\mu(mg)h)(t)| =$$
$$|(m\mu(f)h)(t) - (m\mu(g)h)(t)| \leq \|m\mu(f)h - m\mu(g)h\| \leq$$
$$\|m\|\|\mu(f)h - \mu(g)h\| = \|m\||f(t)h(t) - g(t)h(t)| = 0.$$

We define ϕ by $\phi(t) = mf(t)$ where f is any function in $C_0(T)$ of norm 1 such that $f(t) = 1$. Since we can choose the same f for each point s lying in a neighbourhood of t and mf is continuous, we see that ϕ is continuous. We show that ϕ is bounded by $\|m\|$. Taking h as above we have

$$|\phi(t)| = |mf(t)| = |mf(t)h(t)| = |(\mu(mf)h)(t)| =$$
$$|(m\mu(f)h)(t)| \leq \|m\mu(f)h\| \leq \|m\|\|\mu(f)h\| = \|m\||f(t)| = \|m\|.$$

So, $\phi \in C_b(T)$.

We finally show that $m = \mu(\phi)$. Let $h \in \ell^2(T)$ be of finite support and take $f \in C_0(T)$ such that f equals 1 on the support of h. We then have:

$$\mu(\phi)h = \mu(mf)h = m\mu(f)h = mh.$$

Since the linear span of these fuctions is dense in $\ell^2(T)$ we obtain $m = \mu(\phi)$. So, $m = \mu(\phi) \in \mu(C_b(T))$. $\square$

3. Factorization

Theorem 15. *Let A be C^*-algebra and E a Hilbert C^*-module over A. Let $x \in E$. Then there exists a unique $y \in E$ such that $x = y\langle y, y\rangle$.*

Let A be C^*-algebra and E_1 and E_2 be two Hilbert C^*-module over A. We define the adjointable operators (resp. the compact operators) from E_1 to E_2 taking into account the obvious modifications.

Proposition 16. *Let A be C^*-algebra and E a Hilbert C^*-module over A. We also consider A as a Hilbert C^*-module over A. Let $x \in E$. We set*
$L_x : A \to E$ by $L_x a = xa$
and
$D_x : E \to A$ by $D_x(y) = \langle x, y\rangle$.
Then $L_x \in \mathcal{L}(A, E)$ and $D_x \in \mathcal{L}(E, A)$. Moreover $(L_x)^* = D_x$.

Lemma 17. (1) *The map $x \to D_x$ is an isometric conjugate linear isomorphism from E onto $\mathcal{K}(E, A)$.*
(2) *The map $x \to L_x$ is a is an isometric linear isomorphism from E onto $\mathcal{K}(A, E)$.*

Proof. (1) We have $\|D_x\| = \sup\{\| \langle x, y\rangle \| : \|y\| \leq 1\} = \|x\|$, and hence D is isometric. The image of D is then closed. We also have for $x \in E$ and $a \in A$ that $\Theta_{a,x} = D_{xa^*}$. It follows that the image of D contains $\mathcal{K}(E, A)$. On the other hand, for $x \in E$ the operator D_x is in $\mathcal{K}(E, A)$ since D is continuous and EA is dense in E.
(2) Note that $L_x = D_x^*$ and apply the first part.

$\square$

Let A be C^*-algebra and E a Hilbert C^*-module over A. Let $a \in A$, $T \in \mathcal{K}(E, A)$. $S \in \mathcal{K}(A, E)$ and $R \in \mathcal{K}(E)$. Then

$$\begin{bmatrix} a & T \\ S & R \end{bmatrix}$$

defines an operator on the Hilbert C^*-module $A \oplus E$ by the formula:

$$\begin{bmatrix} a & T \\ S & R \end{bmatrix}\begin{bmatrix} b \\ x \end{bmatrix} = \begin{bmatrix} ab + Tx \\ Sb + Rx \end{bmatrix}.$$

Proposition 18. *Let A be C^*-algebra and E a Hilbert C^*-module over A. Let $a \in A$, $T \in \mathcal{K}(E, A)$, $S \in \mathcal{K}(A, E)$ and $R \in \mathcal{K}(E)$. Then*
(1) *The operator*

$$\begin{bmatrix} a & T \\ S & R \end{bmatrix}$$

is in $\mathcal{L}(A \oplus E)$ and its adjoint is

$$\begin{bmatrix} a^* & S^* \\ T^* & R^* \end{bmatrix}.$$

(2) *The operator*

$$\begin{bmatrix} a & T \\ S & R \end{bmatrix}$$

is in $\mathcal{K}(A \oplus E)$.

(3) *Every compact operator* M *on the Hilbert* C^**-module* $A \oplus E$ *is of the form*

$$\begin{bmatrix} a & T \\ S & R \end{bmatrix}$$

for some $a \in A$, $T \in \mathcal{K}(E, A)$, $S \in \mathcal{K}(A, E)$ *and* $R \in \mathcal{K}(E)$.

Proof of Theorem Let $x \in E$. The operator

$$\begin{bmatrix} 0 & D_x \\ L_x & 0 \end{bmatrix}$$

is a self-adjoint operator in $\mathcal{K}(A \oplus E)$ and satisfies

$$\begin{bmatrix} 0 & D_x \\ L_x & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 0 & D_x \\ L_x & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

It follows from Lemma 17 and Proposition 18 that any self-adjoint operator

$$\begin{bmatrix} a & T \\ S & R \end{bmatrix}$$

in $\mathcal{K}(A \oplus E)$ which satisfies

$$\begin{bmatrix} a & T \\ S & R \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} a & T \\ S & R \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

is of the form

$$\begin{bmatrix} 0 & D_y \\ L_y & 0 \end{bmatrix}$$

for some $y \in E$.

If f is the function $f(t) = t^{\frac{1}{3}}$, it follows from functional calculus that the operator

$$f\left(\begin{bmatrix} 0 & D_x \\ L_x & 0 \end{bmatrix}\right)$$

is in $\mathcal{K}(A \oplus E)$ and satisfies

$$f\left(\begin{bmatrix} 0 & D_x \\ L_x & 0 \end{bmatrix}\right)\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} f\left(\begin{bmatrix} 0 & D_x \\ L_x & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Hence

$$f\left(\begin{bmatrix} 0 & D_x \\ L_x & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & D_y \\ L_y & 0 \end{bmatrix}$$

for some $y \in E$.

We obtain

$$\begin{bmatrix} 0 & D_x \\ L_x & 0 \end{bmatrix} = \begin{bmatrix} 0 & D_y \\ L_y & 0 \end{bmatrix}^3 \Rightarrow$$

$$\begin{bmatrix} 0 & D_x \\ L_x & 0 \end{bmatrix} = \begin{bmatrix} 0 & D_y L_y D_y \\ L_y D_y L_y & 0 \end{bmatrix}.$$

Let $b \in A$. We have

$$L_y D_y L_y(b) = L_y D_y(yb) = L_y(\langle y, yb \rangle) = y \langle y, yb \rangle = y \langle y, y \rangle b$$

and also

$$L_x(b) = xb.$$

Hence $x = y \langle y, y \rangle$.

$\square$

References

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