Hilbert C*-modules
Introductory lecture: rough notes

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Motivating Examples [1]

Example 1 Consider $E = M_{23}(\mathbb{C})$. It is a Banach space (with the norm that a matrix x has as an operator $x : \mathbb{C}^3 \to \mathbb{C}^2$) but the product xy makes no sense. What does make sense is the product x^*y, but this does not lie in $M_{23}(\mathbb{C})$; it lies in $A = M_{33}(\mathbb{C})$.

Thus we have a sesquilinear map $E \times E \to A : (x, y) \to x^*y$. Note that we can recover the norm on E from the norm of A via this map: $\|x\|_E = \|x^*x\|_A^{1/2}$.

Also, A *acts on* E on the right: we have a map $E \times A \to E : (x, a) \to xa$.

Example 2 let E be the space of continuous functions $x = (x_1, x_2) : [0, 1] \to \mathbb{C}^2$ and let A be the algebra $C([0, 1])$. We may define a sesquilinear map $E \times E \to A : (x, y) \to \langle x, y\rangle$ where $\langle x, y\rangle (t) = \bar{x}_1(t)y_1(t) + \bar{x}_1(t)y_1(t)$. This time we *define* the norm on E from the norm of A via this map: $\|x\|_E = \|\langle x, x\rangle\|_\infty^{1/2}$.

Also, A *acts on* E by pointwise multiplication: we have a map $E \times A \to E : (x, a) \to xa$ (where $(xa)(t) = (x_1(t)a(t), x_2(t)a(t))$.

Roughly speaking, a *Hilbert C*-module* E over a C* algebra A is a right A-module equipped with an A-valued 'inner product' which is compatible with the action of A on E and which is used to define a complete norm on E.

Definition 1 **(a)** *A* **Banach algebra** *is a complex algebra A equiped with a complete such that $\|ab\| \leqslant \|a\| \|b\|$ for all $a, b \in A$.*
(b) *A* **C*-algebra** *A is a Banach algebra A equiped with an involution*[2] *$a \to a^*$ satisfying the* **C*-condition**

$$\|a^*a\| = \|a\|^2 \qquad \textit{for all } \ a \in A.$$

The basic example is the algebra $\mathcal{B}(\mathcal{H})$ of all bounded operators on a Hilbert space $\mathcal{H}$ equipped with the operator norm

$$\|a\| = \sup\{\|a\xi\|_{\mathcal{H}} : \xi \in \mathcal{H}, \|\xi\|_{\mathcal{H}} = 1\}$$

[1] hmod, October 15, 2012

[2] that is, a map $A \to A$ such that $(a + \lambda b)^* = a^* + \bar{\lambda}b^*$, $(ab)^* = b^*a^*$, $a^{**} = a$ for all $a, b \in A$ and $\lambda \in \mathbb{C}$

where the involution is defined by $\langle a^*\xi, \eta\rangle = \langle \xi, a\eta\rangle$.

Any C*-algebra can be considered as [3] a closed subalgebra of $\mathcal{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$ (the Gelfand - Naimark theorem).

Any *abelian* C*-algebra is isometrically *-isomorphic to an algebra of the form $C_0(X)$, the continuous complex-valued functions on a locally compact Hausdorff space X, equipped with pointwise operations and the supremum norm.

An element $a \in A$ is called *positive* if $\langle a\xi, \xi\rangle \geqslant 0$ for all $\xi \in \mathcal{H}$ (here A is considered to be a subalgebra of $\mathcal{B}(\mathcal{H})$) [4]. An element $a \in A$ is positive if and only if it is of the form $a = b^*b$ for some $b \in A$. In particular, a positive element a is selfadjoint, i.e. $a = a^*$. An element $a \in C_0(X)$ is positive if and only if $a(t) \in \mathbb{R}_+$ for all $t \in X$.

The set A_+ of positive elements of a C*-algebra A is a closed convex cone, and the norm is monotone on A_+: if $0 \leqslant a \leqslant b$ (meaning that $b - a \in A_+$) then $\|a\| \leqslant \|b\|$. [5]

Definition 2 *Let A be a C^*-algebra. An* **inner product A-module** *is a complex vector space E such that*
(a) E is a right A-module, i.e. there is a bilinear map

$$E \times A \to A : (x, a) \to x \cdot a$$

satisfying $(x \cdot a) \cdot b = x \cdot (ab)$ and $(\lambda x) \cdot a = x \cdot (\lambda a)$ (and $x \cdot \mathbf{1} = x$ when A has a unit $\mathbf{1}$).
(b)There is a map

$$E \times E \to A : (x, y) \to \langle x, y\rangle$$

satisfying

1. $\langle x, \lambda y + z\rangle = \lambda\langle x, y\rangle + \langle x, y\rangle$
2. $\langle x, y \cdot a\rangle = \langle x, y\rangle a$
3. $\langle x, y\rangle^* = \langle y, x\rangle$
4. $\langle x, x\rangle \in A_+$
5. $\langle x, x\rangle = 0 \Rightarrow x = 0 \qquad (x, y, z \in E,\ a \in A,\ \lambda \in \mathbb{C})$.

If only conditions (1) to (4) hold, E is called a **semi-inner product A-module***.*

We prove the Cauchy - Schwarz inequality. We will need the simple observation that $\langle xa, y\rangle = (\langle y, xa\rangle)^* = (\langle y, x\rangle a)^* = a^* \langle x, y\rangle$.

[3] more accurately, is isometrically *-isomorphic to
[4] this is not the original definition of positivity, but it is equivalent to it
[5] Indeed if $a = c^*c$ and $b = d^*d$ the inequality $a \leqslant b$ gives $\|c\xi\| \leqslant \|d\xi\|$ for all $\xi \in \mathcal{H}$ and so $\|c\| \leqslant \|d\|$ hence $\|a\| \leqslant \|b\|$.

Proposition 3 *In a semi-inner product A-module E, for all $x, y \in E$,*

$$\begin{aligned} & \langle y, x\rangle \langle x, y\rangle \leqslant \|\langle x, x\rangle\|_A \langle y, y\rangle \\ \textit{hence} \qquad & \|\langle x, y\rangle\|_A^2 \leqslant \|\langle x, x\rangle\|_A \|\langle y, y\rangle\|_A \end{aligned}$$

Proof. For $a = \langle x, y\rangle \in A$ and all $t \in \mathbb{R}$, we have

$$\begin{aligned} 0 \leqslant \langle xa - ty, xa - ty\rangle &= \langle xa, xa\rangle - t\langle xa, y\rangle - t\langle y, xa\rangle + t^2 \langle y, y\rangle \\ &= a^* \langle x, x\rangle a - ta^*a - ta^*a + t^2 \langle y, y\rangle \\ \text{hence} \quad 2ta^*a &\leqslant a^* \|\langle x, x\rangle\|_A a + t^2 \langle y, y\rangle \qquad (*) \end{aligned}$$

(we have used the inequality $a^*ba \leqslant a^* \|b\| a$, for $b \geqslant 0$). [6] If $\langle x, x\rangle = 0$, the previous inequality gives, for all $t > 0$

$$0 \leqslant 2ta^*a \leqslant t^2 \langle y, y\rangle \quad \Rightarrow \quad 0 \leqslant 2a^*a \leqslant t\langle y, y\rangle .$$

Letting $t \to 0$, this forces $a^*a = 0$ and the Cauchy- Schwarz inequality holds. If $\langle x, x\rangle \neq 0$, setting $t = \|\langle x, x\rangle\|_A$, we get from (*)

$$ta^*a \leqslant t^2 \langle y, y\rangle \quad \text{or} \quad a^*a \leqslant \|\langle x, x\rangle\|_A \langle y, y\rangle$$

which is the required first inequality.

For the second inequality, since $0 \leqslant \langle y, x\rangle \langle x, y\rangle \leqslant \|\langle x, x\rangle\|_A \langle y, y\rangle$, monotonicity of the norm on A_+ gives

$$\begin{aligned} & \left\|\langle x, y\rangle^* \langle x, y\rangle\right\|_A \leqslant \left\| \|\langle x, x\rangle\|_A \langle y, y\rangle \right\|_A \\ \text{so} \quad & \|\langle x, y\rangle\|_A^2 \leqslant \|\langle x, x\rangle\|_A \|\langle y, y\rangle\|_A . \qquad \square \end{aligned}$$

Definition 3 *If E is a semi-inner product A-module, we write*

$$\|x\|_E := \|\langle x, x\rangle\|_A^{1/2} \qquad (x \in E).$$

This is a seminorm on E; [7] it is a norm iff E is an inner product A-module.
A **Hilbert C*-module over** *A is an inner product A-module such that $(E, \|\cdot\|_E)$ is complete.*

Remark 4 *If E is a semi-inner A-module, then by the last Proposition the set $N = \{x \in E : \|x\|_E = 0\}$ is a closed subspace and an A-submodule (if $x \in N$ then $x \cdot a \in N$) and so the quotient E/N becomes an inner product A-module in the induced operations.*

[6] Proof: assuming $A \subseteq \mathcal{B}(\mathcal{H})$, for all $\xi \in \mathcal{H}$ we have $(a^*ba\xi, \xi) = (b(a\xi), (a\xi)) \geqslant 0$ and so $(a^*ba\xi, \xi) = (b(a\xi), (a\xi)) \leqslant \|b\| \|a\xi\|^2 = \|b\| (a\xi, a\xi) = (a^* \|b\| a\xi, \xi)$.

[7] $\|\langle x+y, x+y\rangle\|_A \leqslant \|\langle x, x\rangle\|_A + \|\langle x, y\rangle\|_A + \|\langle y, x\rangle_A\| + \|\langle y, y\rangle\|_A \leqslant \|\langle x, x\rangle\|_A + 2\sqrt{\|\langle x, x\rangle\|_A \|\langle y, y\rangle\|_A} + \|\langle y, y\rangle\|_A$

Corollary 5 *If E is an inner product A-module, the map $E \times A \to E : (x, a) \to x \cdot a$ is continuous; in fact*

$$\|x \cdot a\|_E \leqslant \|x\|_E \|a\|_A .$$

It follows that for each $a \in A$ tha map $R_a : E \to E : x \to x \cdot a$ is bounded with $\|R_a\| \leqslant \|a\|_A$. Thus the map $R : a \to R_a$ is a contractive anti-homomorphism of the Banach algebra A into the Banach algebra $\mathcal{B}(E)$.

Proposition 6 *The closed linear span $\overline{[EA]}$ of the set $\{x \cdot a : x \in E, a \in A\}$ is the whole of E. We say that E is a* non-degenerate *A-module.*

Proof. If A contains a unit $\mathbf{1}$, this is obvious from the relation $x \cdot \mathbf{1} = x$: in fact $E = EA$ in this case. For the general case, we use the fact that every C*-algebra contains a *positive contractive approximate identity*, i.e. a net of contractions (e_i) in A_+ such that for all $a \in A$ we have $\lim_i \|ae_i - a\| = 0$ (hence also $\lim_i \|e_i a - a\| = 0$). Thus every $x \in E$ can be approximated by $xe_i \in EA$. Indeed,

$$\begin{aligned} \|x - xe_i\|_E^2 &= \|\langle x - xe_i, x - xe_i\rangle\|_A = \|\langle x, x\rangle - \langle x, xe_i\rangle + \langle xe_i, xe_i\rangle - \langle xe_i, x\rangle\|_A \\ &= \|\langle x, x\rangle - \langle x, x\rangle e_i + e_i \langle x, x\rangle e_i - e_i \langle x, x\rangle\|_A \\ &\leqslant \|\langle x, x\rangle - \langle x, x\rangle e_i\|_A + \|e_i(\langle x, x\rangle e_i - \langle x, x\rangle)\|_A \\ &\leqslant \|\langle x, x\rangle - \langle x, x\rangle e_i\|_A + \|\langle x, x\rangle e_i - \langle x, x\rangle\|_A \to 0. \qquad \square \end{aligned}$$

Examples

- Any C*-algebra A is a Hilbert C*-module over A with $\langle a, b\rangle = a^* b$ and $a \cdot b = ab$.
- Any closed right ideal J of A is an A-submodule, hence a Hilbert C*-module over A.
- Any Hilbert space $\mathcal{H}$ is a Hilbert C*-module over $\mathbb{C}$.
- But $\mathcal{H}$ it is also a Hilbert C*-module E over $\mathcal{B}(\mathcal{H})$. It is clearer to see this for $\mathcal{H} = \ell^2$: We identify each $x \in \ell^2$ with the $1 \times \infty$ matrix $[x_1, x_2, \ldots]$, i.e. with the linear operator $f_x : \eta \to (\sum_n x_n \eta_n) : \mathcal{H} \to \mathbb{C}$. The module action is given by matrix multiplication: for $a = [a_{ij}] \in \mathcal{B}(\mathcal{H})$, the element $x \cdot a$ is the row matrix $[\sum_i x_i a_{i1}, \sum_i x_i a_{i2}, \ldots]$. Finally, the $\mathcal{B}(\mathcal{H})$-valued inner product is $\langle x, y\rangle = [\bar{x}_i y_j]$. The Hilbert C*-module norm $\|x\|_E$ actually coincides with the ℓ^2 norm of x. [8]
- The *direct sum* $\bigoplus_{k=1}^n E_k$ of finitely many Hilbert C*-modules over *the same* C*-algebra A is the vector space direct sum equipped with coordinate-wise inner product and module action:

$$\langle (x_k), (y_k)\rangle_E := \sum_{k=1}^n \langle x(k), x(k)\rangle_{E_k} \quad \text{and} \quad (x_k) \cdot a := (x_k \cdot a).$$

Example 7 *The direct sum $\bigoplus E_k$ of a sequence of Hilbert C*-modules over a fixed C*-algebra A is defined to be*

$$E = \bigoplus E_k := \{x = (x(k)) \in \prod_k E_k : \sum_k \langle x(k), x(k)\rangle_{E_k} \text{ converges in the norm of } A\}.$$

[8] This can be thought as a $1 \times \infty$ version of the 2×3 example 1.

We prove that E has the properties of a Hilbert C*-modules over A:

Linear space structure Suppose $(x(k)),(y(k)) \in E$; we show that $(x(k)+y(k)) \in E$: Let

$$a_n = \sum_{k \leqslant n} \langle x(k), x(k) \rangle, \quad b_n = \sum_{k \leqslant n} \langle y(k), y(k \rangle, \quad c_n = \sum_{k \leqslant n} \langle x(k)+y(k), x(k)+y(k) \rangle$$

these are in A_+. Since

$$0 \leqslant \langle x+y, x+y \rangle \leqslant \langle x+y, x+y \rangle + \langle x-y, x-y \rangle = 2\langle x, x \rangle + 2\langle y, y \rangle$$

when $n \geqslant m$ the differences $c_n - c_m$ etc. are finite sums of such terms and hence

$$0 \leqslant c_n - c_m \leqslant 2(a_n - a_m) + 2(b_n - b_m) \quad \Rightarrow \quad \|c_n - c_m\|_A \leqslant 2\,\|(a_n - a_m) + (b_n - b_m)\|_A$$

(monotonicity of norm). Since $(x(k))$ and $(y(k))$ are in E, the sequences (a_n) and (b_n) converge in A; hence (c_n) is Cauchy in A, showing that $(x(k)+y(k))$ belongs to E.

Module action Define $(x(k)) \cdot a := (x(k)a)$. This maps E to E because

$$\sum_{k=m}^{n} \langle x(k)a, x(k)a \rangle = \sum_{k=m}^{n} a^* \langle x(k), x(k)a \rangle a = a^* \left(\sum_{k=m}^{n} \langle x(k), x(k) \rangle \right) a$$

which shows that when $\sum_k \langle x(k), x(k) \rangle$ converges in A, so does $\sum_k \langle x(k)a, x(k)a \rangle$.

***A*-valued product**

$$\text{Define} \qquad \langle (x(k)),(y(k)) \rangle_E := \sum_k \langle x(k), y(k) \rangle_{E_k}$$

This series converges in A by polarization: we have

$$\begin{aligned} 4\sum_k \langle x(k), y(k) \rangle &= \sum_k \langle x(k)+y(k), x(k)+y(k) \rangle - \sum_k \langle x(k)-y(k), x(k)-y(k) \rangle \\ &\quad + i\sum_k \langle x(k)+iy(k), x(k)+iy(k) \rangle - i\sum_k \langle x(k)-iy(k), x(k)-iy(k) \rangle \end{aligned}$$

and since $x + i^m y \in E\,(m = 0,1,2,3)$, the four series on the right converge in A, hence so does the one on the left.

Norm This is of course given by

$$\|(x(k))\|_E^2 = \|\langle (x(k)),(x(k)) \rangle_E\|_A = \left\| \sum_k \langle x(k), y(k) \rangle_{E_k} \right\|_A .$$

To prove completeness, we need a simple observation:

Remark If $(x(k)) \in E$ then for each k, since

$$0 \leqslant \langle x(k), x(k) \rangle \leqslant \sum_m \langle x(m), x(m) \rangle \quad \text{in } \; A_+$$

$$\text{we have} \quad \|x(k)\|_{E_k}^2 = \|\langle x(k), x(k) \rangle\|_A \leqslant \left\| \sum_m \langle x(m), x(m) \rangle \right\|_A = \|(x(k))\|_E^2$$

by monotonicity of the norm. It follows that the coordinate projection $Q_k : E \to E_k$ which is clearly an A-module map, is contractive.

Completeness Suppose (x_n) is a $\|\cdot\|_E$-Cauchy sequence, where each $x_n = (x_n(k))$ with $x_n(k) \in E_k$. By the last remark, for each fixed k, the sequence $(x_n(k))_n$ is Cauchy in E_k, hence convergent. Thus there exists $y(k) \in E_k$ with $\lim_n \|y(k) - x_n(k)\|_{E_k} = 0$.

Let $y = (y(k)) \in \prod E_k$. We need to prove two things:
(a) that $y \in E$ and
(b) that $\lim_n \|y - x_n\|_E = 0$.

(a) Proof that $y \in E$, i.e. that $\sum_k \langle y(k), y(k)\rangle$ converges in A.

Since A is complete, we need to show that this series satisfies the Cauchy criterion: given $\varepsilon > 0$ we need to prove that there exists $P \in \mathbb{N}$ such that

$$m \geqslant n \geqslant P \implies \left\| \sum_{k=n}^{m} \langle y(k), y(k)\rangle \right\|_A \leqslant \varepsilon^2. \tag{1}$$

Let us use the notation

$$\|z\|_{n,m} := \left\| \sum_{k=n}^{m} \langle z(k), z(k)\rangle \right\|_A^{1/2}$$

for all $z = (z(k)) \in \prod E_k$ and $m \geqslant n$.

Since (x_n) is Cauchy, given $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$k \geqslant l \geqslant N \implies \|x_k - x_l\|_E < \varepsilon/3. \tag{2}$$

Now $x_N \in E$, so $\sum_i \langle x_N(i), x_N(i)\rangle$ converges in A; hence we can choose $P \geqslant N$ such that

$$\left\| \sum_{i=P}^{\infty} \langle x_N(i), x_N(i)\rangle \right\|_A^{1/2} < \varepsilon/3. \tag{3}$$

Now for each $k \in \mathbb{N}$ we have $y(k) = \lim_M x_M(k)$ in E_k, i.e. $\lim_M \|\langle y(k) - x_M(k), y(k) - x_M(k)\rangle\|_A = 0$. Hence for each $m \geqslant n$ we have

$$\|y - x_M\|_{n,m}^2 = \left\| \sum_{k=n}^{m} \langle y(k) - x_M(k), y(k) - x_M(k)\rangle \right\|_A \leqslant \sum_{k=n}^{m} \|\langle y(k) - x_M(k), y(k) - x_M(k)\rangle\|_A \longrightarrow_{M\to\infty} 0$$

and therefore we may choose $M \geqslant P$ (depending on m, n) such that

$$\|y - x_M\|_{n,m} < \varepsilon/3. \tag{4}$$

Thus if $m \geqslant n \geqslant P$ we have

$$\begin{aligned} \|y\|_{n,m} &\leqslant \|y - x_M\|_{n,m} + \|x_M - x_N\|_{n,m} + \|x_N\|_{n,m} \\ &\leqslant \|y - x_M\|_{n,m} + \|x_M - x_N\|_E + \left\| \sum_{i=n}^{m} \langle x_N(i), x_N(i)\rangle \right\|_A^{1/2} \end{aligned}$$

where we have used the inequality $\|x_M - x_N\|_{n,m} \leqslant \|x_M - x_N\|_E$. [9] The first term is $< \varepsilon/3$ by (4); the second is $< \varepsilon/3$ by (2) because $M \geqslant N$; and the third term is $< \varepsilon/3$ by (3) because $n \geqslant P$. Therefore

$$\|y\|_{n,m} < \varepsilon$$

which proves (1). Hence $y \in E$.

(b) Proof that $\lim_n \|y - x_n\|_E = 0$.

Given $\varepsilon > 0$, let $n_0 \in \mathbb{N}$ be such that

$$n, m \geqslant n_0 \implies \|x_n - x_m\|_E < \varepsilon.$$

Then, if $n, m \geqslant n_0$, we have for all $N \in \mathbb{N}$

$$\left\| \sum_{k=1}^{N} \langle x_n(k) - x_m(k), x_n(k) - x_m(k) \rangle \right\|_A < \varepsilon^2.$$

Letting $m \to \infty$, since each $x_m(k) \to y(k)$ in E_k and there are finitely many terms, we obtain

$$\left\| \sum_{k=1}^{N} \langle x_n(k) - y(k), x_n(k) - y(k) \rangle \right\|_A \leqslant \varepsilon^2$$

for all $n \geqslant n_0$. But now we know from part (a) that $x_n - y \in E$ and so the series

$$\sum_{k=1}^{\infty} \langle x_n(k) - y(k), x_n(k) - y(k) \rangle$$

converges in A_+. Thus if we set

$$a_N = \sum_{k=1}^{N} \langle x_n(k) - y(k), x_n(k) - y(k) \rangle \quad \text{and} \quad a = \sum_{k=1}^{\infty} \langle x_n(k) - y(k), x_n(k) - y(k) \rangle$$

then $\|a_N - a\|_A = 0$ and so $\|a\|_A = \lim_N \|a_N\|_A$. Since each $\|a_N\|_A \leqslant \varepsilon^2$, it follows that $\|a\|_A \leqslant \varepsilon^2$. But this says precisely that $\|y - x_n\|_E \leqslant \varepsilon$ for all $n \geqslant n_0$, as required. $\square$

Remark 8 *On the defining condition for the norm of $x = (x(k))$:*

$$\|(x(k))\|_E^2 = \|\langle (x(k)), (x(k)) \rangle_E\|_A = \left\| \sum_k \langle x(k), y(k) \rangle_{E_k} \right\|_A .$$

[9] Indeed for all $z \in E$ we have

$$0 \leqslant \sum_{k=n}^{m} \langle z(k), z(k) \rangle \leqslant \sum_{k=1}^{\infty} \langle z(k), z(k) \rangle \quad \text{in } A_+, \quad \text{hence} \quad \|z\|_{n,m}^2 \leqslant \|z\|_E^2$$

by monotonicity of the norm.

Since the partial sums $a_n = \sum_{k \leqslant n} \langle x(k), y(k) \rangle_{E_k}$ form an increasing sequence in A_+ such that $\lim_n \|a_n - a\|_A = 0$ where $a = \sum_{k=1}^{\infty} \langle x(k), y(k) \rangle_{E_k}$, we have

$$\|(x(k))\|_E^2 = \|a\|_A = \lim_n \|a_n\|_A = \lim_n \left\| \sum_{k \leqslant n} \langle x(k), y(k) \rangle \right\|_A$$

Note that IF the series converges *absolutely*, i.e. if we require that $\sum_{k=1}^{\infty} \|\langle x(k), y(k) \rangle\|_A < \infty$, then certainly $(x(k)) \in E$; but this is too strong a condition. [10]

On the other hand, if we merely assume that the partial sums $\sum_{k \leqslant n} \langle x(k), y(k) \rangle$ are bounded above, then the series *need not converge in the norm of* A although it does converge strongly (but its limit need not be in A); for example we may take each $E_k = A =$ the compact operators, and $(x(k))$ to be a sequence of orthogonal rank one projections: the sum is not compact.

A special case of the direct sum is partcularly important for the theory.

Definition 4 *The* standard C*-module *over a C*-algebra A, sometimes denoted $\mathcal{H}_A$, is the direct sum $\bigoplus E_K$, where each E_k equals the Hilbert C*-module A. Thus*

$$\mathcal{H}_A := \{x = (x(k)) : \textit{each } x(k) \in A \textit{ and } \sum_k x(k)^* x(k) \textit{ converges in the norm of } A\}.$$

Thus, in case $A = \mathbb{C}$, the standard module is just $\ell^2(\mathbb{N})$.

References

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[10] Exercise: Find a sequence (f_k) of elements of $A = c_0$ such that $\sum_k |f_k|^2$ converges in the norm of A, but $\sum_k \|f_k\|_\infty^2 = +\infty$. Can you do the same in the algebra $C([0,1)]$?