A NOTE ON CUNTZ ALGEBRAS

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Let J be the set of all pairs of isometries [1] (S_{1j}, S_{2j}) acting on a separable Hilbert space H_j satisfying the *Cuntz relation*

$$S_{1j}S_{1j}^* + S_{2j}S_{2j}^* = I. \tag{C}$$

In other words, S_{1j} and S_{2j} are isometries (i.e. their initial projections $S_{1j}^*S_{1j}$ and $S_{2j}^*S_{2j}$ are equal to the identity) whose final (or range) projections $P_{1j} = S_{1j}S_{1j}^*$ and $P_{2j} = S_{2j}S_{2j}^*$ are orthogonal and sum to the identity I. Such isometries exist on any infinite dimensional Hilbert space.

For example, on $L^2([0,1])$ we may define S_1 to be the natural isometry onto $L^2([0,1/2])$ given by $(S_1f)(x) = \sqrt{2}f(2x)$, while S_2 maps onto $L^2([1/2,1])$ and is given by $(S_2f)(x) = \sqrt{2}f(2x-1)$. (In fact, one could consider n-tuples rather than pairs, but we stick to the case $n = 2$ for simplicity of notation.)

Consider the Hilbert space $H = \oplus_j H_j$ and the direct sums

$$S_1 = \oplus_j S_{1j} \quad \text{and} \quad S_2 = \oplus_j S_{2j}$$

acting on H. Thus, H consists of all functions $j \to x_j$ in $\prod_j H_j$ which are square summable and the operator S_1 is defined on H by $S_1(x_j) = (S_{1j}x_j)$ (this is well-defined since $\sum \|S_{1j}x_j\|^2 \leqslant \sum \|x_j\|^2$); similarly $S_2(x_j) = (S_{2j}x_j)$. It is clear that S_1 and S_2 are isometries and that they satisfy (C).

Let $\mathcal{O}_2$ be the closed *-subalgebra of $B(H)$ generated by S_1 and S_2. By construction, $\mathcal{O}_2$ has the following

Universal Property. *For any pair (T_1, T_2) of isometries acting on a separable Hilbert space K which satisfy (C), there exists a (necessarily contractive) unique *-homomorphism π mapping $\mathcal{O}_2$ onto the C^*-algebra $C^*(T_1, T_2)$ generated by the pair (T_1, T_2) such that $\pi(S_1) = T_1$ and $\pi(S_2) = T_2$.*

Let $\mathcal{A} \subseteq \mathcal{O}_2$ be the unital *-algebra which is linearly generated by all words in the generators S_1, S_2 and their adjoints. If $\mu = i_1 \dots i_k$ where $i_j \in \{1,2\}$, we write $S_\mu = S_{i_1} \dots S_{i_k}$, where $|\mu| = k$ is the length of the word. We claim that any word in S_1, S_2 and their adjoints may be written in the form $S_\mu S_\lambda^*$. This follows from

Lemma 1. *Suppose $S_\mu^* S_\lambda \neq 0$.*

(i) If $|\mu| = |\lambda|$, then $\mu = \lambda$ and $S_\mu^* S_\lambda = I$.

(ii) If $|\mu| < |\lambda|$, then there is a word ν such that $\lambda = \mu\nu$ and $S_\mu^* S_\lambda = S_\nu$.

(iii) If $|\mu| > |\lambda|$, then there is a word ν' such that $\mu = \nu'\lambda$ and $S_\mu^* S_\lambda = S_{\nu'}^*$.

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Proof. If $\mu = \lambda$ then $S_\mu^* S_\lambda = S_{i_k}^* \dots (S_{i_1}^* S_{i_1}) \dots S_{i_k} = I$. If $\lambda \neq \mu$, and are of equal length, if i_j is the first digit of λ which differs from the corresponding digit i'_j in μ, then $S_{i'_j}^* S_{i_j} = 0$ and so $S_\mu^* S_\lambda = S_{i_k}^* \dots (S_{i'_j}^* S_{i_j}) \dots S_{i_k} = 0$.

The other assertions follow similarly. □

The gauge automorphisms of $\mathcal{O}_2$. For $|z| = 1$, we define γ_z as follows. We first put

$$\gamma_z(S_i) = zS_i \quad \gamma_z(S_i^*) = \bar{z}S_i^*, \quad (i = 1, 2).$$

Thus for example $\gamma_z(S_1) = \oplus_j zS_{1j}$. The map γ_z uniquely defines a *-automorphism of the unital *-algebra $\mathcal{A} \subseteq \mathcal{O}_2$. On monomials, γ_z is given by

$$\gamma_z(S_\mu S_\lambda^*) = z^{|\mu|} S_\mu z^{-|\lambda|} S_\lambda^*$$

and it extends linearly to$\mathcal{A}$.

We claim that each γ_z extends to an automorphism of $\mathcal{O}_2$. For this, fix a $z \in \mathbb{T}$. Given $j \in J$, consider the pair (zS_{1j}, zS_{2j}). Observe that it is a pair of separably acting isometries satisfying (C); since J consists of *all* such pairs, there must exist some $j(z) \in J$ such that $(zS_{1j}, zS_{2j}) = (S_{1j(z)}, S_{2j(z)})$. Since $(\bar{z}S_{1j(z)}, \bar{z}S_{2j(z)}) = (S_{1j}, S_{2j})$, the map $j \to j(z)$ is onto; in other words, z defines a permutation $j \to j(z)$ of J.

Noting that each $a \in \mathcal{A} \subseteq \mathcal{O}_2$ is a direct sum $\oplus a_j$ where $a_j \in B(H_j)$ is in the unital *-algebra generated by (S_{1j}, S_{2j}), we see that γ_z will map the j-th summand a_j of a to the $j(z)$-th summand $a_{j(z)}$; that is, γ_z just permutes the summands of a:

$$\gamma_z(a) = \gamma_z(\oplus a_j) = \oplus a_{j(z)}.$$

It therefore follows that

$$\|\gamma_z(a)\| = \sup_j \left\|a_{j(z)}\right\| = \sup_j \|a_j\| = \|a\|.$$

Thus γ_z is an isometry of $\mathcal{A}$ for every $z \in \mathbb{T}$ and hence extends to an isometry of $\mathcal{O}_2$ to itself (which we denote again by γ_z). Since its range contains $\mathcal{A}$, the extended map is onto $\mathcal{O}_2$; finally, since γ_z is a *-homomorphism on the dense *-subalgebra $\mathcal{A}$, it will be a *-homomorphism on the whole of $\mathcal{O}_2$.

Conclusion: For each $z \in \mathbb{T}$ there is a *-automorphism γ_z of $\mathcal{O}_2$ uniquely specified by $\gamma_z(S_i) = zS_i \, (i = 1, 2)$. Of course the map $z \to \gamma_z$ is a group homomorphism.

By the Lemma,

$$\mathcal{A} = \mathrm{span}\{S_\mu S_\lambda^* : \text{all } \mu, \lambda\} \quad \text{hence} \quad \mathcal{O}_2 = \overline{\mathrm{span}}\{S_\mu S_\lambda^* : \text{all } \mu, \lambda\}.$$

Define $\mathcal{F}_0 = \mathrm{span}\{S_\mu S_\lambda^* : |\mu| = |\lambda|\}$ and $\mathcal{F} = \overline{\mathcal{F}_0}$.

Now $\mathcal{F}_0 = \bigcup_{k \in \mathbb{N}} \mathcal{F}_k$ where $\mathcal{F}_k = \mathrm{span}\{S_\mu S_\lambda^* : |\mu| = |\lambda| = k\}$. For example $\mathcal{F}_1 = \mathrm{span}\{S_1 S_1^*, S_1 S_2^*, S_2 S_1^*, S_2 S_2^*\}$ is *-isomorphic to $M_2(\mathbb{C})$ via the map $S_i S_j^* \to E_{i,j}$. More generally $\mathcal{F}_k \simeq M_{n^k}(\mathbb{C})$. In fact $\mathcal{F}_k$ embeds as a unital subalgebra of $\mathcal{F}_{k+1}$ [2] and so $\mathcal{F}$ is a UHF (uniformly hyperfinite) C^*-subalgebra of $\mathcal{O}_2$.

[2] for example $S_1 S_2^* = S_1(S_1 S_1^* + S_2 S_2^*) S_2^* = S_1 S_1 S_1^* S_1^* + S_1 S_2 S_2^* S_2^* \in \mathrm{span}\{S_\mu S_\lambda^* : |\mu| = |\lambda| = 2\}$

The conditional expectation. Note that for each $a \in \mathcal{O}_2$ the map

$$f_a : [0, 2\pi] \to \mathcal{O}_2 : t \to \gamma_{e^{it}}(a)$$

is continuous, hence Riemann integrable. We define

$$E_0(a) = \frac{1}{2\pi} \int_0^{2\pi} f_a(t) dt$$

and we note that $\|E_0(a)\| \leqslant \|a\|$. It is easy to see that, since $\gamma_z(S_\mu S_\lambda^*) = z^{|\mu|-|\lambda|} S_\mu S_\lambda^*$, we have

$$E_0(S_\mu S_\lambda^*) = \begin{cases} 0, & |\mu| \neq |\lambda| \\ S_\mu S_\lambda^*, & |\mu| = |\lambda| \end{cases}$$

It follows that E_0 maps $\mathcal{A} = \text{span}\{S_\mu S_\lambda^* : \text{all } \mu, \lambda\}$ onto $\mathcal{F}_0 = \text{span}\{S_\mu S_\lambda^* : |\mu| = |\lambda|\}$ and is the identity map on $\mathcal{F}_0$. By continuity, E_0 maps $\mathcal{O}_2$ (contractively) into $\mathcal{F}$ and is the identity on $\mathcal{F}$. Thus it is a unital contractive idempotent of $\mathcal{O}_2$ onto $\mathcal{F}$. Finally, if $a = b^*b$ is a positive nonzero element of $\mathcal{O}_2$, then $\gamma_z(a) = \gamma_z(b)^*\gamma_z(b)$ is positive and nonzero for all $z \in \mathbb{T}$, and hence the integral $E_0(a)$ does not vanish.

Conclusion: E_0 is a *faithful* conditional expectation.

Simplicity of $\mathcal{O}_2$. Let $a \in \mathcal{O}_2$ and $b \in \mathcal{F}$. Since $\gamma_z(ab) = \gamma_z(a)\gamma_z(b) = \gamma_z(a)b$, it follows that $E_0(ab) = E_0(a)b$ and similarly $E_0(ba) = bE_0(a)$.

It follows that if $\mathcal{J} \subseteq \mathcal{O}_2$ is a (closed, two-sided) ideal, then the closure $\mathcal{J}_0$ of $E_0(\mathcal{J})$ is an ideal of $\mathcal{F}$; and if $\mathcal{J}$ contains a nonzero element a, then it will contain a nonzero positive element a^*a, hence $\mathcal{J}_0$ will contain the nonzero positive element $b = E_0(a^*a)$ and will thus be nontrivial.

Then for each $k \in \mathbb{N}$ the intersection $\mathcal{J}_0 \cap \mathcal{F}_k$ will be an ideal $\mathcal{J}_k$ of $\mathcal{F}_k$. But recall that $\mathcal{F}_k$ is isomorphic to $M_{n^k}(\mathbb{C})$ which is simple. Thus either some intersection $\mathcal{J}_k$ contains the identity of $\mathcal{F}_k$, which is the identity of $\mathcal{O}_2$, or else all the intersections are $\{0\}$. In the first case the ideal $\mathcal{J}$ contains the identity hence $\mathcal{J} = \mathcal{O}_2$. In the second case, for each k the map

$$\mathcal{F}_k \to \mathcal{F}/\mathcal{J}_0 : x \to x + \mathcal{J}_0$$

is a *-homomorphism and it is injective because $\mathcal{J}_0 \cap \mathcal{F}_k = \{0\}$. But $\mathcal{F}/\mathcal{J}_0$ and $\mathcal{F}_k$ are C*-algebras, hence the above map is an isometry. Thus $\|x + \mathcal{J}_0\| = \text{dist}(x, \mathcal{J}_0) = \|x\|$ for all $x \in \mathcal{F}_k$. But $b \in \mathcal{F} = \overline{\cup \mathcal{F}_k}$ and so there are $b_k \in \mathcal{F}_k$ with $\|b_k - b\| \to 0$. But then

$$\|b_k\| = \text{dist}(b_k, \mathcal{J}_0) \leqslant \|b_k - b\| \to 0$$

which forces $b = \lim b_k = 0$.

Therefore $\mathcal{O}_2$ cannot have nontrivial ideals.

Conclusion: Any *-representation of $\mathcal{O}_2$ is necessarily faithful (i.e. 1-1), because its kernel is an ideal. Therefore, *any pair* T_1, T_2 of isometries satisfying the Cuntz relation (C) generates a C*-algebra $C^*(T_1, T_2)$ which is isomorphic to $\mathcal{O}_2$.

... and there are many interesting such representations, on separable Hilbert space.