Operator Systems in Quantum Contextuality and Nonlocality

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Outline

Nonlocality

Fix A, B, X, Y finite sets.

Alice's lab:

Questions: X
Answers: A
Measurements: $\{E_{a,x}\}_{a\in A}$, $x \in X$

Bob's lab:

Questions: Y
Answers: B
Measurements: $\{F_{b,y}\}_{b\in B}$, $y \in Y$

$$\text{Correlations} \rightsquigarrow p = \{(p(a,b|x,y))_{a,b} : x, y\}$$

Local correlations: Convex combinations of $p_A(a|x) \cdot p_B(b|y)$. Notation: $\mathcal{C}_{loc}$.

Quantum: Assuming the tensor paradigm $p(a,b|x,y) = \langle E_{a,x} \otimes F_{y,b}\psi, \psi\rangle$, with

$$\psi \in H_A \otimes H_B, \quad \{E_{a,x}\}_{a\in A} \subseteq \mathcal{B}(H_A), \quad \{F_{b,y}\}_{b\in B} \subseteq \mathcal{B}(H_B) \text{ POVM's.}$$

*We assume H_A, H_B finite dimensional. Notation: $\mathcal{C}_q$.

Quantum commuting: Assuming the commutativity paradigm

$$p(a,b|x,y) = \langle E_{a,x} F_{y,b} \psi, \psi \rangle \text{ such that}$$
$$\psi \in H, \quad \{E_{a,x}\}_{a \in A}, \{F_{b,y}\}_{b \in B} \subseteq \mathcal{B}(H) \text{ POVM's}, \quad E_{a,x} F_{b,y} = F_{b,y} E_{a,x}.$$

Notation: $\mathcal{C}_{qc}$.

$$\mathcal{C}_{loc} \subseteq \mathcal{C}_q \subseteq \mathcal{C}_{qc} \, .$$

Nonlocality: Correlations p with $p \in \mathcal{C}_q \setminus \mathcal{C}_{loc}$ (Bell's Theorem, CHSH inequality)

Tsirelson's Problem (TP): Is $\overline{\mathcal{C}_q} = \mathcal{C}_{qc}$? (No, MIP*=RE 20')

We denote $\mathcal{C}_{qa} := \overline{\mathcal{C}_q}$.

Connes, Tsirelson, and Kirchberg's problems

Kirchberg's Problem (KP): Is $C^*(\mathbb{F}_2) \otimes_{min} C^*(\mathbb{F}_2) = C^*(\mathbb{F}_2) \otimes_{max} C^*(\mathbb{F}_2)$?

Tsirelson's Problem $\Leftrightarrow$ Kirchberg's Problem $\Leftrightarrow$ Connes Embedding Problem

KP $\Rightarrow$ TP : Passes through the following characterisation

Theorem [Fritz 10']: Set $\mathbb{F}_{X,A} = \underbrace{\mathbb{Z}_A * \cdots * \mathbb{Z}_A}_{X-times}$ (similarly $\mathbb{F}_{Y,B}$). A correlation p is in the set:

1. $\mathcal{C}_{qa}$ if and only if there exists a state s of $C^*(\mathbb{F}_{X,A}) \otimes_{min} C^*(\mathbb{F}_{Y,B})$ such that

$$p(a, b|x, y) = s(e_{x,a} \otimes e_{y,b})$$

2. $\mathcal{C}_{qc}$ if and only if there exists a state s of $C^*(\mathbb{F}_{X,A}) \otimes_{max} C^*(\mathbb{F}_{Y,B})$ such that

$$p(a, b|x, y) = s(e_{x,a} \otimes e_{y,b})$$

via operator systems

Set $\mathcal{A}_{X,A} = \underbrace{\ell^\infty_A *_1 \cdots *_1 \ell^\infty_A}_{X-times}$ and $\mathcal{S}_{X,A} = \underbrace{\ell^\infty_A \oplus_1 \cdots \oplus_1 \ell^\infty_A}_{X-times}$ where $\mathcal{S}_{X,A} \subseteq \mathcal{A}_{X,A}$.

Using $C^*(\mathbb{F}_{X,A}) = \mathcal{A}_{X,A}$ and the theory of tensor products for operator systems:

<u>**Theorem**</u> [Paulsen-Todorov 13']: A correlation p is in the set:

1. $\mathcal{C}_{qa}$ if and only if there exists a state s of $\mathcal{S}_{X,A} \otimes_{min} \mathcal{S}_{Y,B}$ such that

$$p(a,b|x,y) = s(e_{x,a} \otimes e_{y,b})$$

2. $\mathcal{C}_{qc}$ if and only if there exists a state s of $\mathcal{S}_{X,A} \otimes_{c} \mathcal{S}_{Y,B}$ such that

$$p(a,b|x,y) = s(e_{x,a} \otimes e_{y,b})$$

Outline

We denote the algebra of bounded operators on a Hilbert space H by $B(H)$.

Definition: A **concrete operator system** is a unital self-adjoint subspace $\mathcal{S} \subseteq B(H)$, meaning:

$$\mathcal{S}^* = \mathcal{S}, \quad I_H \in \mathcal{S}.$$

Definition: An **abstract operator system** is a $*$-vector space $\mathcal{S}$ equipped with:

- a matrix ordering $\{C_n\}_{n\in\mathbb{N}}$ (cones $C_n \subseteq M_n(\mathcal{S})_{\mathrm{sa}}$), and
- an **Archimedean matrix order unit** $e \in \mathcal{S}$.

Definition: Let $\mathcal{S}, \mathcal{T}$ be operator systems. A linear map $\phi : \mathcal{S} \to \mathcal{T}$ is **unital completely positive** (u.c.p.) if

$$\phi^{(n)} : M_n(\mathcal{S}) \to M_n(\mathcal{T}), \quad [s_{ij}] \mapsto [\phi(s_{ij})]$$

is positive for all n and $\phi(e_{\mathcal{S}}) = e_{\mathcal{T}}$. We say that ϕ is a **complete order isomorphism (c.o.i.)**, if ϕ is a completely positive bijection and ϕ^{-1} is completely positive and a **complete order embedding (c.o.e.)**, if ϕ is a complete order isomorphism onto its range.

Choi–Effros Representation Theorem: The concrete and abstract definitions of operator systems coincide.

States, Extensions, and Dilations

- A *state* of an operator system $\mathcal{S}$ is a unital positive linear functional.

Arveson's Extension Theorem: Let $\mathcal{S} \subseteq \mathcal{T}$ be operator systems and $\phi : \mathcal{S} \to \mathcal{B}(H)$ a unital completely positive map. Then there exists a u.c.p. map $\tilde{\phi} : \mathcal{T} \to \mathcal{B}(H)$ such that $\tilde{\phi}|_{\mathcal{S}} = \phi$ and $\|\tilde{\phi}\| = \|\phi\|$.

Stinespring's Dilation Theorem: Let $\mathcal{A}$ be a unital C^*-algebra and $\phi : \mathcal{A} \to \mathcal{B}(H)$ a unital completely positive map. Then there exists a Hilbert space K, an isometry $V : H \to K$, and a unital *-homomorphism $\pi : \mathcal{A} \to \mathcal{B}(K)$ such that

$$\phi(x) = V^*\pi(x)V, \quad \forall x \in \mathcal{A}.$$

- If $\phi(x) = V^*\pi(x)V$ with V an isometry, and H is identified with $V(H)$, then $\phi(x) = P_H\pi(x)|_H$, i.e., ϕ is the compression of a *-homomorphism.
- Given an operator system $\mathcal{S}$, a C^*-cover is a pair $(\mathcal{C}, \iota)$, where $\mathcal{C}$ is a unital C^*-algebra and $\iota : \mathcal{S} \to \mathcal{C}$ is a unital complete order embedding such that $\iota(\mathcal{S})$ generates $\mathcal{C}$ as a C^*-algebra.

Tensor Products of C*-Algebras

Let A, B be unital C*-algebras.

- There is a minimal and a maximal C*-algebra tensor product, denoted by $A \otimes_{\mathsf{min}} B$ and $A \otimes_{\mathsf{max}} B$ respectively.

- For any appropriate norm $\| \cdot \|_\gamma$ on $A \otimes B$ that turns its completion into a C*-algebra we have:

$$\|x\|_{\mathsf{min}} \leq \|x\|_\gamma \leq \|x\|_{\mathsf{max}}$$

- A C*-algebra A is **nuclear** if

$$A \otimes_{\mathsf{min}} B = A \otimes_{\mathsf{max}} B \quad \forall \text{ C*-algebras } B$$

Remark: The minimal tensor product is injective: If $A_0 \subseteq A$, $B_0 \subseteq B \implies$

$$A_0 \otimes_{\min} B_0 \subseteq A \otimes_{\min} B.$$

Operator System Tensor Products

Let $\mathcal{S}, \mathcal{T}$ be operator systems.

- An **operator system structure** on $\mathcal{S} \otimes \mathcal{T}$ is a family of cones $\{C_n^\tau\}_{n=1}^\infty \subseteq M_n(\mathcal{S} \otimes \mathcal{T})$ satisfying some reasonable properties such that

$$\mathcal{S} \otimes_\tau \mathcal{T} := (\mathcal{S} \otimes \mathcal{T}, \{C_n^\tau\}, e_\mathcal{S} \otimes e_\mathcal{T}) \text{ is an operator system.}$$

- We may write $C_n = M_n(\mathcal{S} \otimes_\tau \mathcal{T})^+$ and given two structures τ_1, τ_2, we write $\tau_1 \geq \tau_2$ if

$$M_n(\mathcal{S} \otimes_{\tau_1} \mathcal{T})^+ \subseteq M_n(\mathcal{S} \otimes_{\tau_2} \mathcal{T})^+$$

- An **operator system tensor product** is a map τ assigning to each pair $(\mathcal{S}, \mathcal{T})$ a structure $\mathcal{S} \otimes_\tau \mathcal{T}$.

- Let α and β be two operator system tensor products. An operator system $\mathcal{S}$ is called (α, β)-**nuclear** if for every operator system $\mathcal{T}$:

$$\mathcal{S} \otimes_\alpha \mathcal{T} \cong \mathcal{S} \otimes_\beta \mathcal{T}$$

Minimal and maximal tensor products

Minimal tensor product:

For $\mathcal{S} \subseteq \mathcal{B}(H)$, $\mathcal{T} \subseteq \mathcal{B}(K)$, then

$$\mathcal{S} \otimes_{\min} \mathcal{T} \subseteq \mathcal{B}(H \otimes K)$$

Maximal tensor product:

$$D_n^{\max} = \left\{ \alpha(P \otimes Q)\alpha^* \in M_n(\mathcal{S} \otimes \mathcal{T}) : P \in M_k(\mathcal{S})^+, Q \in M_m(\mathcal{T})^+, \alpha \in M_{n,km} \right\}$$

$$C_n^{\max} = \{A \in M_n(\mathcal{S} \otimes \mathcal{T}) : re_n + A \in D_n^{\max} \text{ for all } r > 0\}$$

$$\mathcal{S} \otimes_{\max} \mathcal{T} := (\mathcal{S} \otimes \mathcal{T}, \{C_n^{\max}\}, e_1 \otimes e_2)$$

Commuting and Essential tensor product

We have two extremal C*-covers:

- The C^*-**envelope** $C_e^*(\mathcal{S})$ is the unique C^*-cover having the following universal property: For any C^*-cover $\iota : \mathcal{S} \hookrightarrow \mathcal{A}$ there exists a unique unital *-homomorphism $\pi : \mathcal{A} \to C_e^*(\mathcal{S})$ such that $\pi(\iota(s)) = s$ for every $s \in \mathcal{S}$.
- The **universal** C^*-cover is the unique C^*-algebra $C_u^*(\mathcal{S})$ generated by $\mathcal{S}$ such that for any other C^*-algebra $\mathcal{B}$ and unital completely positive map $\phi : \mathcal{S} \to \mathcal{B}$, there exists a *-homomorphism $\pi_\phi : C_u^*(\mathcal{S}) \to \mathcal{B}$ that extends ϕ.

Commuting tensor product:

$$\mathcal{S} \otimes_{\mathsf{c}} \mathcal{T} \subseteq C_u^*(\mathcal{S}) \otimes_{\max} C_u^*(\mathcal{T})$$

Essential tensor product:

$$\mathcal{S} \otimes_{\mathsf{ess}} \mathcal{T} \subseteq C_e^*(\mathcal{S}) \otimes_{\max} C_e^*(\mathcal{T})$$

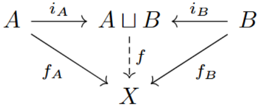
$$\text{min} \leq \text{ess} \leq \text{c} \leq \text{max}$$

Coproducts

Definition: In a category $\mathcal{C}$, a **coproduct** of objects A and B is an object $A \sqcup B$ together with morphisms

$$i_A : A \to A \sqcup B, \quad i_B : B \to A \sqcup B$$

such that for any object X and morphisms $f_A : A \to X$, $f_B : B \to X$, there exists a unique morphism $f : A \sqcup B \to X$ making the following diagram commute:



Unital C*-algebras: The coproduct $A *_1 B$ identifies the units of A and B. More generally, if $\mathcal{D} \subseteq A, B$, then $A *_{\mathcal{D}} B$ amalgamates over $\mathcal{D}$.

Operator systems: The coproduct $\mathcal{S} \oplus_1 \mathcal{T}$ is the universal operator system for unital completely positive maps from $\mathcal{S}$ and $\mathcal{T}$.

Operator systems encode measurements

Let X, A be finite sets.

- ℓ_A^∞ encodes POVM's:

$$\{E_a\}_{a\in A} \text{ POVM on } H \longleftrightarrow \phi : \ell_A^\infty \to \mathcal{B}(H) : \phi(\delta_a) = E_a \text{ is ucp.}$$

- $\mathcal{S}_{X,A} := \underbrace{\ell_A^\infty \oplus_1 \cdots \oplus_1 \ell_A^\infty}_{|X|-times}$ encodes families of POVM's:

$$\{E_{a,x}\}_{a\in A} \text{ POVM on } H, \forall x \in X \longleftrightarrow \phi : \mathcal{S}_{X,A} \to \mathcal{B}(H) : \phi(\delta_{a,x}) = E_{a,x} \text{ is ucp}$$

where $\{\delta_{a,x}\}_{a\in A}$ is the canonical basis of the x-th copy of ℓ_A^∞.

Motivation: The measurements $\{E_{x,a}\}_{a\in A}, x \in X$ considered are disjoint. We want to encode measurements with shared entries (e.g. in contextuality).

Outline

Operator $\mathcal{A}$-systems

Definition: Let $\mathcal{S}$ be an operator system and $\mathcal{A}$ a unital C*-algebra. We say that $\mathcal{S}$ is an (abstract) **operator $\mathcal{A}$-system** if:

1. $\mathcal{S}$ is an $\mathcal{A}$-bimodule
2. $(a \cdot s)^* = s^* \cdot a^*$
3. $a \cdot e = e \cdot a$
4. $[a_{i,j}] \cdot [s_{i,j}] \cdot [a_{i,j}]^* \in M_n(\mathcal{S})^+$

for all $[a_{i,j}] \in M_{n,m}(\mathcal{A})$, $[s_{i,j}] \in M_m(\mathcal{S})^+$, $s \in \mathcal{S}$, $a \in \mathcal{A}$.

Concretely: Suppose

$$1 \in \mathcal{A} \subseteq \mathcal{S} \subseteq \mathcal{B}(H),$$

where $\mathcal{S}$ is a concrete operator system and $\mathcal{A}$ is a C*-algebra such that the operator multiplication satisfies $\mathcal{A} \cdot \mathcal{S} \subseteq \mathcal{S}$. Then

$$\mathcal{S} \cdot \mathcal{A} = \mathcal{S}^* \cdot \mathcal{A}^* = (\mathcal{A} \cdot \mathcal{S})^* \subseteq \mathcal{S}^* = \mathcal{S},$$

and hence $\mathcal{S}$ is an operator $\mathcal{A}$-system.

Representation Theorem for Operator $\mathcal{A}$-systems

Theorem: Let $\mathcal{A}$ be a unital C^*-algebra and $\mathcal{S}$ an operator $\mathcal{A}$-system. Then there exist a Hilbert space H, a unital complete order embedding $\phi : \mathcal{S} \to \mathcal{B}(H)$, and a unital $*$-homomorphism $\pi : \mathcal{A} \to \mathcal{B}(H)$ such that

$$\phi(a \cdot s) = \pi(a)\,\phi(s), \quad \forall a \in \mathcal{A},\ s \in \mathcal{S}\,.$$

- Consider the category whose objects are operator $\mathcal{A}$-systems and whose morphisms are unital completely positive (ucp) $\mathcal{A}$-bimodule maps; that is, maps

$$\phi : \mathcal{S} \to \mathcal{T}$$

that satisfy:

- ϕ is unital and completely positive,
- $\phi(a \cdot s) = a \cdot \phi(s)$ and $\phi(s \cdot a) = \phi(s) \cdot a$, for all $a \in \mathcal{A}, s \in \mathcal{S}$.

Remark: Operator systems = operator $\mathcal{A}$-systems with $\mathcal{A} = \mathbb{C}$.

No coproducts in the category

- Let $\mathcal{A} = \mathbb{C} \oplus \mathbb{C}$, and define operator $\mathcal{A}$-systems

$$\mathcal{S} = \mathbb{C} \oplus 0, \quad \mathcal{T} = 0 \oplus \mathbb{C}$$

with actions

$$a \cdot s = a_1 s, \quad a \cdot t = a_2 t, \quad a = (a_1, a_2) \in \mathcal{A}\,.$$

- Assume that X is a coproduct with ucp $\mathcal{A}$-bimodule maps $\phi_1 : \mathcal{S} \to X$, $\phi_2 : \mathcal{T} \to X$.

- For $a' = (0, a_2)$,

$$a' \cdot e_{\mathcal{S}} = 0 \implies a' \cdot e_X = 0.$$

Similarly, for $a'' = (a_1, 0)$,

$$a'' \cdot e_{\mathcal{T}} = 0 \implies a'' \cdot e_X = 0.$$

- For all $a \in \mathcal{A}$,

$$a \cdot e_X = 0,$$

but $1_{\mathcal{A}} \cdot e_X = e_X \neq 0$.

Conclusion: No coproduct exists.

Faithful Operator $\mathcal{A}$-Systems

Definition: Let $\mathcal{S}$ be an operator $\mathcal{A}$-system with module action $a \cdot s$. We say $\mathcal{S}$ is **faithful** if

$$a \cdot e \neq 0, \quad \text{for all } a \in \mathcal{A} \setminus \{0\}.$$

Remark: If $\mathcal{S}$ is faithful, then by the representation Theorem there exist

$$H, \quad \phi : \mathcal{S} \to \mathcal{B}(H) \text{ (unital c.o.e.)}, \quad \pi : \mathcal{A} \to \mathcal{B}(H) \text{ (faithful *-rep.)}$$

such that

$$\phi(a \cdot s) = \pi(a)\, \phi(s), \quad \forall a \in \mathcal{A}, s \in \mathcal{S}\,.$$

We can then identify

$$1 \in \mathcal{A} \subseteq \mathcal{S} \subseteq \mathcal{B}(H)$$

and view the module action as operator multiplication.

Coproducts of Faithful Operator $\mathcal{A}$-Systems

Theorem [*C.*]: Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be faithful operator $\mathcal{A}$-systems. Then their coproduct exists in the category of operator $\mathcal{A}$-systems with morphisms the ucp $\mathcal{A}$-bimodule maps. Moreover:

- The coproduct is itself a faithful operator $\mathcal{A}$-system.
- It is unique up to a complete order isomorphism that is also an $\mathcal{A}$-bimodule map.

Proof (sketch)

- Let $\mathcal{S}_1 \subseteq \mathcal{B}(H_1)$ and $\mathcal{S}_2 \subseteq \mathcal{B}(H_2)$ be faithful operator $\mathcal{A}$-systems with faithful *-representations $\pi_i : \mathcal{A} \to \mathcal{B}(H_i)$.
- Consider the amalgamated free product $\mathcal{B}(H_1) *_{\mathcal{A}} \mathcal{B}(H_2)$, the universal C^*-algebra amalgamated over $\mathcal{A}$.
- Define the operator system:

$$\mathcal{S}_1 + \mathcal{S}_2 := \{s_1 + s_2 : s_i \in \mathcal{S}_i\} \subseteq \mathcal{B}(H_1) *_{\mathcal{A}} \mathcal{B}(H_2).$$

- $\mathcal{S}_1 + \mathcal{S}_2$ is an operator $\mathcal{A}$-system with module action given by multiplication inside the free product.

Universal Property

- Given an operator $\mathcal{A}$-system $\mathcal{T} \subseteq \mathcal{B}(K)$ and unital completely positive $\mathcal{A}$-bimodule maps
$$\psi_i : \mathcal{S}_i \to \mathcal{T},$$
extend them via Arveson's extension theorem to
$$\tilde{\psi}_i : \mathcal{B}(H_i) \to \mathcal{B}(K),$$
agreeing on $\mathcal{A}$.
- By Boca's theorem, there exists a unital completely positive map
$$\Psi : \mathcal{B}(H_1) *_{\mathcal{A}} \mathcal{B}(H_2) \to \mathcal{B}(K),$$
extending $\tilde{\psi}_i$.
- Restricting Ψ to $\mathcal{S}_1 + \mathcal{S}_2$ yields a unital completely positive $\mathcal{A}$-bimodule map Φ with
$$\Phi|_{\mathcal{S}_i} = \psi_i.$$
- Hence, $\mathcal{S}_1 + \mathcal{S}_2$ satisfies the universal property of the coproduct.

Kernel Realization of the Coproduct

Theorem [C.]: Let $\mathcal{A}$ be a unital C^*-algebra, and let $\mathcal{S}, \mathcal{T}$ be faithful operator $\mathcal{A}$-systems. Define the subspace

$$\mathcal{J} := \{a \oplus (-a) : a \in \mathcal{A}\} \subseteq \mathcal{S} \oplus \mathcal{T}\,.$$

Then:

- The quotient $\mathcal{S} \oplus \mathcal{T}\,/\mathcal{J}$ admits an operator $\mathcal{A}$-system structure.
- There is a complete order isomorphism

$$\mathcal{S} \oplus_{\mathcal{A}} \mathcal{T} \cong \mathcal{S} \oplus \mathcal{T}\,/\mathcal{J},$$

which preserves the $\mathcal{A}$-bimodule structure.

$\mathcal{S} \oplus_{\mathcal{A}} \mathcal{T}$ denotes the amalgamated coproduct.

Graph Operator Systems as Operator $\mathcal{A}$-Systems

Definition: Let $G = (V, E)$ be a graph on n vertices. Define an operator system $\mathcal{S}_G \subseteq M_n$ as

$$\mathcal{S}_G = \mathsf{span}\{E_{i,j} : i = j \text{ or } (i,j) \in E\}.$$

This is called the *graph operator system* of G.

- Each $\mathcal{S}_G$ is a $\mathcal{D}_n$-bimodule, so it is a faithful operator $\mathcal{D}_n$-system.
- Conversely, any operator subsystem of M_n that is a $\mathcal{D}_n$-bimodule arises this way from a graph G with

$$E = \{(i,j) : i \neq j \text{ and } E_{i,i}\,\mathcal{S}\,E_{j,j} \neq \{0\}\}.$$

Question: Is the coproduct of two graph operator systems itself a graph operator system?

Arveson's hyperrigidity

Unique Extension Property. Let $\mathcal{S}$ be an operator system and $(\mathcal{A}, \iota)$ a C^*-cover of $\mathcal{S}$. A unital completely positive map $\phi : \mathcal{S} \to \mathcal{B}(H)$ has the *unique extension property* with respect to $(\mathcal{A}, \iota)$ if it extends uniquely to a completely positive map $\widetilde{\phi} : \mathcal{A} \to \mathcal{B}(H)$ that is also a *-representation.

Definition (Hyperrigidity). Let $\mathcal{S} \subseteq \mathcal{A}$ be an operator system and $(\mathcal{A}, \iota)$ a C^*-cover. $\mathcal{S}$ is *hyperrigid* in $\mathcal{A}$ if for every representation $\pi : \mathcal{A} \to \mathcal{B}(H)$, the restriction $\pi|_{\mathcal{S}}$ has the unique extension property.

Proposition. If $\mathcal{S}$ is hyperrigid in $\mathcal{A}$, then $\mathcal{A} \cong C_e^*(\mathcal{S})$.

Theorem [*C.*]: Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be faithful operator $\mathcal{A}$-systems that are hyperrigid in their respective C^*-envelopes. Then:

$$C_e^*(\mathcal{S}_1 \oplus_{\mathcal{A}} \mathcal{S}_2) \cong C_e^*(\mathcal{S}_1) *_{\mathcal{A}} C_e^*(\mathcal{S}_2).$$

Proposition [*C.*]: The coproduct of two graph operator systems is not necessarily a graph operator system.

There exist graph operator systems whose coproduct is not completely order isomorphic to any operator system $\mathcal{S}_{G'} \subseteq M_k$ that is a bimodule over $\mathcal{D}_k$ for some $k \in \mathbb{N}$.

Outline of the proof:

- Let G be the complete graph on 2 vertices; then $\mathcal{S}_G = M_2$.
- Consider the coproduct $\mathcal{S}_G \oplus_{\mathcal{D}_2} \mathcal{S}_G = M_2 \oplus_{\mathcal{D}_2} M_2$.
- Assume this is completely order isomorphic to a graph operator system $\mathcal{S}_{G'} \subseteq M_k$.
- M_2 is hyperrigid
- From the previous theorem

$$M_2 *_{\mathcal{D}_2} M_2 \cong C_e^*(M_2 \oplus_{\mathcal{D}_2} M_2) \cong C_e^*(\mathcal{S}_{G'}) \cong C^*(\mathcal{S}_{G'})/\mathcal{I}.$$

- But $M_2 *_{\mathcal{D}_2} M_2$ is infinite-dimensional (contains words of any length).
- Contradiction as $C^*(\mathcal{S}_{G'}) \subseteq M_k$

Outline

Introduction

A hypergraph is a pair $\mathbb{G} = (V, E)$, where V is a finite set and E is a finite set of subsets of V.

<u>Definition</u>: A **contextuality scenario** is a hypergraph $\mathbb{G} = (V, E)$ such that $\cup_{e \in E} e = V$.

Vertices represent the "outcomes" and edges represent the "measurements".

<u>Definition</u>: Let $\mathbb{G} = (V, E)$ be a contextuality scenario. A **probabilistic model** on $\mathbb{G}$, is an assignment $p : V \to [0, 1]$ such that

$$\sum_{x \in e} p(x) = 1, \quad \text{for every} \quad e \in E.$$

<u>Notation</u>: $\mathcal{G}(\mathbb{G})$.

*Not all scenarios admit probabilistic models. We restrict to the ones that do.

• This hypergraph theoretic framework was introduced by *A. Acín, T. Fritz, A. Leverrier, A. B. Sainz 15'* to study contextuality.

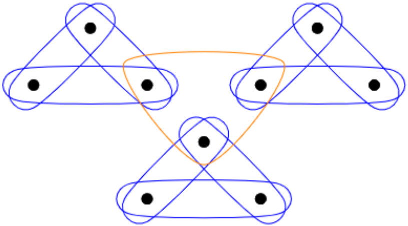


Figure 1: Example of a scenario that does not admit a probabilistic model.

Definition: Let $\mathbb{G} = (V, E)$ be a contextuality scenario. A **Projective Representation (PR)** of $\mathbb{G}$ on a Hilbert space H is a collection of projections $(P_x)_{x \in V} \subseteq \mathcal{B}(H)$ such that $\sum_{x \in e} P_x = 1$, for every $e \in E$.

Consider the scenario $\mathbb{B}_{X,A}$ such that

$$V = X \times A \text{ and } E = \big\{\{x\} \times A : x \in X\big\},$$

then a PR $E = (E_{x,a})_{x \in X, a \in A}$ is a family of PVM's. Such scenarios are called Bell scenarios.

Definition: Let $\mathbb{G} = (V, E)$ be a contextuality scenario. A probabilistic model $p \in \mathcal{G}(\mathbb{G})$ is called

1. **deterministic**, if $p(x) \in \{0, 1\}$, $\forall x \in V$.
2. **classical**, if it is a convex combination of deterministic ones. Notation: $\mathcal{C}(\mathbb{G})$
3. **quantum**, if there exists a Hilbert space H, a PR $(P_x)_{x \in V}$ on H and a state $\psi \in H$ such that

$$p(x) = \langle P_x \psi, \psi \rangle \quad \forall x \in V$$

Notation: $\mathcal{Q}(\mathbb{G})$

We have the following chain of inclusions:

$$\mathcal{C}(\mathbb{G}) \subseteq \mathcal{Q}(\mathbb{G}) \subseteq \mathcal{G}(\mathbb{G})$$

Theorem [Kochen-Specker]: There exists a contextuality scenario $\mathbb{G}_{KS}$, such that $\mathcal{C}(\mathbb{G}_{KS}) = \emptyset$, while $\mathcal{Q}(\mathbb{G}_{KS}) \neq \emptyset$.

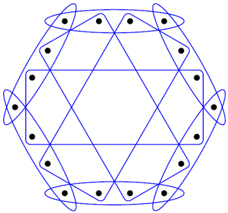
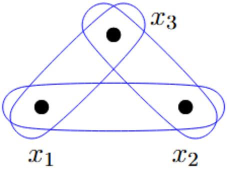


Figure: The scenario $\mathbb{G}_{KS}$ proving the Kochen-Specker Theorem.

- Also, $\mathcal{Q}(\mathbb{G}) \subsetneq \mathcal{G}(\mathbb{G})$



The **free hypergraph C*-algebra** $C^*(\mathbb{G})$ [AFLS15] is the universal C*-algebra generated by orthogonal projections p_x, $x \in V$ such that $\sum_{x \in e} p_x = 1$ for every $e \in E$. e.g. $C^*(\mathbb{B}_{X,A}) = \underbrace{\ell_A^\infty *_1 \cdots *_1 \ell_A^\infty}_{X-times}$.

The *-representations $\pi : C^*(\mathbb{G}) \to \mathcal{B}(H)$ correspond precisely to PR's $(P_x)_{x \in V}$ of $\mathbb{G}$ on H via $\pi(p_x) = P_x$, $x \in V$. Hence quantum models arise as

$$p(x) = \langle \pi(p_x)\xi, \xi \rangle.$$

Our approach

Definition: Let $\mathbb{G} = (V, E)$ be a contextuality scenario. A **Positive Operator Representation (POR)** of $\mathbb{G}$ on a Hilbert space H is a collection $(A_x)_{x \in V} \subseteq \mathcal{B}(H)^+$ such that

$$\sum_{x \in e} A_x = 1, \quad \text{for every} \quad e \in E.$$

A PR, is a POR such that A_x is a projection for every $x \in V$.

- For the Bell scenarios $\mathbb{B}_{X,A}$ a POR $E = (E_{x,a})_{x \in X, a \in A}$ is a family of POVM's.

Remark: A family of POVM's always dilates to a family of PVM's. It's not true for POR's as we will see.

We will construct an operator system universal for positive operator representations.

The operator system for POR's

- Fix a scenario $\mathbb{G} = (V, E)$, and write $E = \{e_1, e_2, \ldots, e_d\}$. For each $e \in E$ we set

$$\mathcal{S} := \ell^{\infty}_{e_1} \oplus \cdots \oplus \ell^{\infty}_{e_d} \,.$$

For $x \in V$, denote by $\delta^e_x \in \ell^{\infty}_e$ the element with 1 in the x-th, and zero in the remaining ones.

- Define

$$\mathcal{J} := \mathrm{span}\{(1 \oplus -1 \oplus \cdots \oplus 0), (1 \oplus 0 \oplus -1 \oplus \cdots \oplus 0), \ldots, (1 \oplus 0 \oplus \cdots \oplus -1), \\ (0 \oplus \cdots \oplus \delta^{e_i}_x \oplus \cdots \oplus -\delta^{e_j}_x \oplus \cdots 0) : \forall i \neq j \in \{1, \ldots, n\} \text{ s.t. } x \in e_i \cap e_j\}.$$

- By taking an appropriate quotient we turn $\mathcal{S} \,/\, \mathcal{J}$ into an operator system.

Remark: If the hyperedges in $\mathbb{G}$ are mutually disjoint, $\mathcal{S} \,/\, \mathcal{J}$ is simply the unital coproduct $\ell^{\infty}_{e_1} \oplus_1 \ell^{\infty}_{e_2} \oplus_1 \cdots \oplus_1 \ell^{\infty}_{e_d}$.

For $e \in E$, let $\iota_e : \ell_e^\infty \to \oplus_{f \in E} \ell_f^\infty$ be the natural embedding let $i_e : \ell_e^\infty \to \mathcal{S} \,/\, \mathcal{J}$ be the map given by

$$i_e(u) = |E|(q \circ \iota_e)(u), \quad u \in \ell_e^\infty.$$

The maps i_e are ucp but may not always be complete order embeddings so set

$$a_x := i_e(\delta_x^e), \quad x \in V$$

and thus $\mathcal{S} \,/\, \mathcal{J} = \mathrm{span}\{a_x : x \in V\}$.

Proposition[*Anoussis, C., Todorov*]: If $\Phi : \mathcal{S} \,/\, \mathcal{J} \to \mathcal{B}(H)$ is a unital completely positive map then $(\Phi(a_x))_{x \in V}$ is a POR of $\mathbb{G}$. Conversely, if $(A_x)_{x \in V} \subseteq \mathcal{B}(H)$ is a POR of $\mathbb{G}$ then there exists a unique unital completely positive map $\Phi : \mathcal{S} \,/\, \mathcal{J} \to \mathcal{B}(H)$ such that $\Phi(a_x) = A_x$, $x \in V$. Moreover, it is the unique operator system with this property.

We set $\mathcal{S}_{\mathbb{G}} := \mathcal{S} \,/\, \mathcal{J}$.

The operator system for dilatable POR's

Recall the **free hypergraph C*-algebra** $C^*(\mathbb{G})$,

e.g. $C^*(\mathbb{B}_{X,A}) = \underbrace{\ell_A^\infty *_1 \cdots *_1 \ell_A^\infty}_{X-times}$ while $\mathcal{S}_{\mathbb{B}_{X,A}} = \underbrace{\ell_A^\infty \oplus_1 \cdots \oplus_1 \ell_A^\infty}_{X-times}$

Consider

$$\mathcal{T}_{\mathbb{G}} := \operatorname{span}\{p_x : x \in V\} \subseteq C^*(\mathbb{G}).$$

- We say that a POR $(A_x)_{x\in V} \subseteq \mathcal{B}(H)$ of $\mathbb{G}$ dilates to a PR, if there exist a Hilbert space $\mathcal{K}$, an isometry $V : H \to \mathcal{K}$ and a PR $(P_x)_{x\in V}$ of $\mathbb{G}$ such that $A_x = V^* P_x V$, $x \in V$.

- $\mathcal{T}_{\mathbb{G}}$ is universal for dilatable POR's.

Definition: We say that a POR $(A_x)_{x\in V}$ is **classically dilatable** if there exists a Hilbert space $\mathcal{K}$ and an isometry $V : H \to \mathcal{K}$ and a PR $(P_x)_{x\in V}$ with commuting entries such that $A_x = V^* P_x V$, $x \in V$.

- We can define an operator system $\mathcal{R}_{\mathbb{G}}$ inside an abelian C*-algebra $\mathcal{D}_{\mathbb{G}}$, which is *universal* for classically dilatable POVM representations.

The following diagram of canonical u.c.p. maps arises from their universal properties:

$$\mathcal{S}_{\mathbb{G}} \xrightarrow{\Phi} \mathcal{T}_{\mathbb{G}} \xrightarrow{\Psi} \mathcal{R}_{\mathbb{G}}$$

As a consequence, we obtain the following correspondence:

Prob. models	$\leftrightsquigarrow$	**States on OpSys**
$\mathcal{G}(\mathbb{G})$	$\leftrightsquigarrow$	$\mathcal{S}_{\mathbb{G}}$
$\mathcal{Q}(\mathbb{G})$	$\leftrightsquigarrow$	$\mathcal{T}_{\mathbb{G}}$
$\mathcal{C}(\mathbb{G})$	$\leftrightsquigarrow$	$\mathcal{R}_{\mathbb{G}}$

Dilations

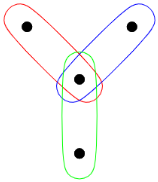
Definition: We say that a scenario $\mathbb{G}$ is dilating (resp. classically dilating), if every POR of $\mathbb{G}$ dilates to a PR (resp. PR with commuting entries) of $\mathbb{G}$.

- e.g. $\mathbb{B}_{X,A}$ is dilating.

Theorem [*Anoussis, C., Todorov*]: Let $\mathbb{G} = (V, E)$ be a contextuality scenario. Then,

- $\mathbb{G}$ is dilating if and only if $\mathcal{S}_{\mathbb{G}} = \mathcal{T}_{\mathbb{G}}$;
- $\mathbb{G}$ is classically dilating if and only if $\mathcal{S}_{\mathbb{G}} = \mathcal{R}_{\mathbb{G}}$

Proposition [*Anoussis, C., Todorov*]: Scenarios $\mathbb{G} = (V, E)$ such that $e' \cap e'' = \bigcap_{e \in E} e \neq \emptyset$ for all $e', e'' \in E$ with $e' \neq e''$ are dilating.



Proof of Proposition:

- To each edge e_j we associate $\ell^\infty_{e_j}$, and view these as (faithful) operator $\mathcal{A}$-systems over the C*-algebra $\mathcal{A}$ generated by vectors of length equal to the size of $f = \bigcap e_j$ in ℓ^∞_V adjoined with a unit.

- A POR corresponds to POVM's of sizes e_j that overlap on f, so they give rise to u.c.p. maps $\phi_{e_j} : \ell^\infty_{e_j} \to \mathcal{B}(H)$ that agree on $\mathcal{A}$.

- By the universal property of the coproduct $\oplus_{\mathcal{A}} \ell^\infty_{e_j}$ we obtain a ucp map $\phi : \oplus_{\mathcal{A}} \ell^\infty_{e_j} \to \mathcal{B}(H)$ which extends to a u.c.p. map Φ on the amalgamated free product of C*-algebras $*_{\mathcal{A}} \ell^\infty_{e_j}$.

- A Stinespring dilation theorem for Φ yields a dilation of the POR into a PR.

Quantum magic squares

<u>Definition</u> $A = [a_{i,j}] \in M_n(\mathcal{B}(\mathbb{C}^s))$ is called a **quantum magic square**, if $a_{i,j} \in \mathcal{B}(\mathbb{C}^s)^+$, $\forall i,j$ and all rows and columns sum to 1. It's called a **quantum permutation matrix** if moreover $a_{i,j}$ are projections.

Given $n \in \mathbb{N}$ define a hypergraph $\mathbb{G}_n$ by

$$V = [n] \times [n] \text{ and } E = \big\{\{i\} \times [n], [n] \times \{j\} : i,j = 1, \ldots, n\big\},$$

so that a quantum magic square $A = [a_{i,j}]_{i,j=1}^n$, is a POR $(a_{i,j})_{(i,j)\in V}$ on $H = \mathbb{C}^s$ (PR if A was a quantum permutation matrix).

[De las Coves, Drescher, Netzer 20']: For every $n \geq 3$ there exists a quantum magic square of size n that does not dilate to a quantum permutation matrix.

Proposition [*Anoussis, C., Todorov*]: For every $n \geq 3$ there is a POR of $\mathbb{G}_n$, that doesn't admit a dilation into a PR. That is, $\mathbb{G}_n$ are not dilating for $n \geq 3$ and $\mathcal{S}_{\mathbb{G}_n} \neq \mathcal{T}_{\mathbb{G}_n}$.

Let $\mathbb{G} = (V, E)$ and $\mathbb{H} = (W, F)$ and $\mathbb{G} \times \mathbb{H} = (V \times W, E \times F)$. A probabilistic model p on $\mathbb{G} \times \mathbb{H}$ is called:

1. deterministic, if $p(x, y) \in \{0, 1\}$ for all $(x, y) \in V \times W$.
2. classical, if it's a convex combination of deterministic models

$$p(x, y) = p^1(x)p^2(y), \;\; x \in V, \; y \in W$$

   where $p^1 \in \mathcal{G}(\mathbb{G})$, $p^2 \in \mathcal{G}(\mathbb{H})$. Notation: $\mathcal{C}(\mathbb{G}, \mathbb{H})$.
3. generalised tensor probabilistic model (resp. tensor probabilistic models), if

$$p(x, y) = \langle (A_x \otimes B_y)\psi, \psi \rangle, \quad (x, y) \in V \times W$$

   for POR's (resp. PR's) $(A_x)_{x \in V} \subseteq \mathcal{B}(H_{\mathbb{G}})$ and $(B_y)_{y \in W} \subseteq \mathcal{B}(H_{\mathbb{H}})$, $\dim H_{\mathbb{G}}$, $\dim H_{\mathbb{H}} < \infty$ and $\psi \in H_{\mathbb{G}} \otimes H_{\mathbb{H}}$ unit vector. Notation: $\tilde{\mathcal{Q}}_q(\mathbb{G}, \mathbb{H})$ (resp. $\mathcal{Q}_q(\mathbb{G}, \mathbb{H})$).
4. generalised commuting probabilistic model (resp. commuting probabilistic models), if

$$p(x, y) = \langle (A_x B_y)\psi, \psi \rangle, \quad (x, y) \in V \times W$$

   for POR's (resp. PR's) $(A_x)_{x \in V} \subseteq \mathcal{B}(H)$ and $(B_y)_{y \in W} \subseteq \mathcal{B}(H)$ that commute and $\psi \in H$ unit vector. Notation: $\tilde{\mathcal{Q}}_{qc}(\mathbb{G}, \mathbb{H})$ (resp. $\mathcal{Q}_{qc}(\mathbb{G}, \mathbb{H})$).

Bell Scenarios and Correlations

A **correlation**

$$p = \{p(a, b \mid x, y)\}_{a \in A,\, b \in B}^{x \in X,\, y \in Y}$$

gives rise to a **probabilistic model** $\tilde{p}$ on $\mathbb{B}_{X,A} \times \mathbb{B}_{Y,B}$ via:

$$p(a, b \mid x, y) \longmapsto \tilde{p}((x, a), (y, b))$$

and vice versa.

Correlations $\leftrightsquigarrow$ **Probabilistic Models**
$\mathcal{C}_{\mathrm{loc}}(X, Y, A, B) = \mathcal{C}(\mathbb{B}_{X,A}, \mathbb{B}_{Y,B})$
$\mathcal{C}_q(X, Y, A, B) = \mathcal{Q}_q(\mathbb{B}_{X,A}, \mathbb{B}_{Y,B})$
$\mathcal{C}_{qa}(X, Y, A, B) = \mathcal{Q}_{qa}(\mathbb{B}_{X,A}, \mathbb{B}_{Y,B})$
$\mathcal{C}_{qc}(X, Y, A, B) = \mathcal{Q}_{qc}(\mathbb{B}_{X,A}, \mathbb{B}_{Y,B})$

Note: $\mathcal{Q}_{qa}(\mathbb{G}, \mathbb{H}) = \overline{\mathcal{Q}_q(\mathbb{G}, \mathbb{H})}$.

Theorem [Anoussis, C., Todorov]:

Prob. Models	$\leftrightsquigarrow$	States on OpSys	$\leftrightsquigarrow$	States on C^*-alg
$\tilde{\mathcal{Q}}_{qc}(\mathbb{G}, \mathbb{H})$	$\leftrightsquigarrow$	$\mathcal{S}_{\mathbb{G}} \otimes_c \mathcal{S}_{\mathbb{H}}$	$\leftrightsquigarrow$	$C_u^*(\mathcal{S}_{\mathbb{G}}) \otimes_{\max} C_u^*(\mathcal{S}_{\mathbb{H}})$
$\tilde{\mathcal{Q}}_{qa}(\mathbb{G}, \mathbb{H})$	$\leftrightsquigarrow$	$\mathcal{S}_{\mathbb{G}} \otimes_{\min} \mathcal{S}_{\mathbb{H}}$	$\leftrightsquigarrow$	$C_u^*(\mathcal{S}_{\mathbb{G}}) \otimes_{\min} C_u^*(\mathcal{S}_{\mathbb{H}})$
$\mathcal{Q}_{qc}(\mathbb{G}, \mathbb{H})$	$\leftrightsquigarrow$	$\mathcal{T}_{\mathbb{G}} \otimes_{\text{ess}} \mathcal{T}_{\mathbb{H}}$	$\leftrightsquigarrow$	$C^*(\mathbb{G}) \otimes_{\max} C^*(\mathbb{H})$
$\mathcal{Q}_{qa}(\mathbb{G}, \mathbb{H})$	$\leftrightsquigarrow$	$\mathcal{T}_{\mathbb{G}} \otimes_{\min} \mathcal{T}_{\mathbb{H}}$	$\leftrightsquigarrow$	$C^*(\mathbb{G}) \otimes_{\min} C^*(\mathbb{H})$
$\mathcal{C}(\mathbb{G}, \mathbb{H})$	$\leftrightsquigarrow$	$\mathcal{R}_{\mathbb{G}} \otimes_{\min} \mathcal{R}_{\mathbb{H}}$	$\leftrightsquigarrow$	$\mathcal{D}_{\mathbb{G}} \otimes_{\min} \mathcal{D}_{\mathbb{H}}$

Where $\tilde{\mathcal{Q}}_{qa}(\mathbb{G}, \mathbb{H}) = \overline{\tilde{\mathcal{Q}}_q(\mathbb{G}, \mathbb{H})}$.

Remarks:

1. $C_u^*(\mathcal{S}_{\mathbb{G}})$ is the universal C^*-cover of $\mathcal{S}_{\mathbb{G}}$ and corresponds also to the universal C^*-algebra generated by positive elements a_x, $x \in V$ that $\sum_{x \in E} a_x = 1$, for all $e \in E$,
2. $C^*(\mathbb{G}) = C_e^*(\mathcal{T}_{\mathbb{G}})$.

Equivalence with Connes embedding problem

Theorem [Anoussis, C., Todorov]: The following are equivalent:

- CEP has an affirmative answer
- $\tilde{\mathcal{Q}}_{\mathrm{qa}}(\mathbb{G}, \mathbb{G}) = \tilde{\mathcal{Q}}_{\mathrm{qc}}(\mathbb{G}, \mathbb{G})$ for every scenario $\mathbb{G}$.
- $C_u^*(\mathcal{S}_{\mathbb{G}}) \otimes_{\min} C_u^*(\mathcal{S}_{\mathbb{G}}) = C_u^*(\mathcal{S}_{\mathbb{G}}) \otimes_{\max} C_u^*(\mathcal{S}_{\mathbb{G}})$ for every scenario $\mathbb{G}$.
- $\mathcal{S}_{\mathbb{G}} \otimes_{\min} \mathcal{S}_{\mathbb{G}} = \mathcal{S}_{\mathbb{G}} \otimes_{\mathrm{c}} \mathcal{S}_{\mathbb{G}}$ for every scenario $\mathbb{G}$.

and also

- CEP has an affirmative answer
- $\mathcal{Q}_{\mathrm{qa}}(\mathbb{G}, \mathbb{G}) = \mathcal{Q}_{\mathrm{qc}}(\mathbb{G}, \mathbb{G})$ for every dilating scenario $\mathbb{G}$.
- $C^*(\mathbb{G}) \otimes_{\min} C^*(\mathbb{G}) = C^*(\mathbb{G}) \otimes_{\max} C^*(\mathbb{G})$ for every dilating scenario $\mathbb{G}$.
- $\mathcal{T}_{\mathbb{G}} \otimes_{\min} \mathcal{T}_{\mathbb{G}} = \mathcal{T}_{\mathbb{G}} \otimes_{\mathrm{c}} \mathcal{T}_{\mathbb{G}}$ for every dilating scenario $\mathbb{G}$.

Equivalence with Connes embedding problem

Theorem [Anoussis, C., Todorov]: The following are equivalent:

- CEP has an affirmative answer
- $\tilde{\mathcal{Q}}_{\rm qa}(\mathbb{G},\mathbb{G}) = \tilde{\mathcal{Q}}_{\rm qc}(\mathbb{G},\mathbb{G})$ for every scenario $\mathbb{G}$.
- $C_u^*(\mathcal{S}_{\mathbb{G}}) \otimes_{\rm min} C_u^*(\mathcal{S}_{\mathbb{G}}) = C_u^*(\mathcal{S}_{\mathbb{G}}) \otimes_{\rm max} C_u^*(\mathcal{S}_{\mathbb{G}})$ for every scenario $\mathbb{G}$.
- $\mathcal{S}_{\mathbb{G}} \otimes_{\rm min} \mathcal{S}_{\mathbb{G}} = \mathcal{S}_{\mathbb{G}} \otimes_{\rm c} \mathcal{S}_{\mathbb{G}}$ for every scenario $\mathbb{G}$.

and also

- CEP has an affirmative answer
- $\mathcal{Q}_{\rm qa}(\mathbb{G},\mathbb{G}) = \mathcal{Q}_{\rm qc}(\mathbb{G},\mathbb{G})$ for every dilating scenario $\mathbb{G}$.
- $C^*(\mathbb{G}) \otimes_{\rm min} C^*(\mathbb{G}) = C^*(\mathbb{G}) \otimes_{\rm max} C^*(\mathbb{G})$ for every dilating scenario $\mathbb{G}$.
- $\mathcal{T}_{\mathbb{G}} \otimes_{\rm min} \mathcal{T}_{\mathbb{G}} = \mathcal{T}_{\mathbb{G}} \otimes_{\rm c} \mathcal{T}_{\mathbb{G}}$ for every dilating scenario $\mathbb{G}$.

Thank you!

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