Boundary Actions and C^*-Algebraic Properties of Discrete Groups

Spyridon Petrakos

National and Kapodistrian University of Athens

April 14, 2022

Notation

Throughout this presentation Γ will denote a *discrete* group.
No assumption on its cardinality is made.

Some Fundamentals

Definition

An *operator system* is a *-closed linear subspace of a unital C^*-algebra containing its unit.

Definition

A linear map φ between operator systems is called *completely positive* (resp. *completely isometric*) iff $\varphi_{(n)} := \varphi \otimes \mathrm{id}_{\mathbb{M}_n(\mathbb{C})}$ is positive (resp. isometric) for all $n \in \mathbb{N}$.

Some Fundamentals

Definition

Γ is called *amenable* iff there exists a state ω on $\ell^\infty(\Gamma)$ which is invariant under the left translation action of Γ, i.e.

$$\omega(sf) = \omega(f)$$

for all $s \in \Gamma$ and $f \in \ell^\infty(\Gamma)$, where $(sf)(t) = f(s^{-1}t)$.

Proposition

There exists a normal amenable subgroup of Γ which contains all other normal amenable subgroups of Γ. We call this subgroup the *amenable radical* of Γ.

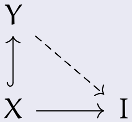
The Idea

dynamical properties of Γ (boundary actions)	$\longleftrightarrow$	C^*-algebraic properties of Γ (properties of $C^*_r(\Gamma)$)
(topological) freeness		C^*-simplicity
faithfulness		unique trace property
amenability		exactness

Γ-Injective Envelopes

Definition

An object I in a category $\mathfrak{C}$ is called *injective* iff every morphism $\mathrm{X} \to \mathrm{I}$ factors through every monomorphism $\mathrm{X} \to \mathrm{Y}$.



We are interested in the category $\Gamma\mathfrak{G}_1$ of Γ-operator systems with Γ-maps as morphisms. Unless otherwise specified, $\mathcal{S}$ will denote an operator system.

Γ-Injective Envelopes

Definition

$\mathcal{S}$ is called a Γ-*operator system* iff Γ acts on it by unital complete order isomorphisms. A Γ-*map* is a Γ-equivariant unital completely positive (u.c.p.) map between Γ-operator systems.

Definition

$\mathcal{S}$ is called Γ-*injective* iff it is injective in $\Gamma\mathfrak{G}_1$.

Definition

A Γ-*extension* of $\mathcal{S}$ is a pair $(\mathcal{T}, \iota)$, where $\mathcal{T}$ is a Γ-operator system and $\iota : \mathcal{S} \to \mathcal{T}$ is a completely isometric Γ-equivariant map.

Γ-Injective Envelopes

Definition

A Γ-extension $(\mathcal{T}, \iota)$ of $\mathcal{S}$ is called:

- Γ-*injective* iff $\mathcal{T}$ is Γ-injective.
- Γ-*essential* iff, for every Γ-map $\varphi : \mathcal{T} \to \mathcal{U}$, φ is completely isometric whenever $\varphi \circ \iota$ is.
- Γ-*rigid* iff, for every Γ-map $\varphi : \mathcal{T} \to \mathcal{T}$, φ is the identity on $\mathcal{T}$ whenever $\varphi \circ \iota = \iota$.

Definition

A Γ-extension of $\mathcal{S}$ that is both Γ-injective and Γ-essential is called a Γ-*injective envelope* of $\mathcal{S}$.

Γ-Injective Envelopes

Theorem (Hamana '85)

Every Γ-operator system $\mathcal{S}$ has a Γ-injective envelope which is unique in the sense that if $(\mathcal{T}_1, \iota_1)$ and $(\mathcal{T}_2, \iota_2)$ are two Γ-injective envelopes of $\mathcal{S}$, then there exists a Γ-isomorphism $\omega : \mathcal{T}_1 \to \mathcal{T}_2$ such that $\omega \circ \iota_1 = \iota_2$.

Γ-injective envelopes are constructed via certain projections, allowing us to endow them with the Choi-Effros product, which turns them into C^*-algebras. We will denote the Γ-injective envelope of $\mathcal{S}$ by $I_\Gamma(\mathcal{S})$.

The Hamana Boundary

Consider $\mathbb{C}$ equipped with the trivial action of Γ. Then, by construction, $\mathrm{I}_\Gamma(\mathbb{C})$ is an abelian C^*-algebra.

Definition

We define the *Hamana Boundary*, $\partial_{\mathrm{H}}\Gamma$, to be the Gelfand spectrum of $\mathrm{I}_\Gamma(\mathbb{C})$.

Topological Boundaries

Definition

A locally compact Hausdorff space is called a Γ-*space* iff Γ acts on it by homeomorphisms.

Note that an action on a compact Γ-space X induces an action on $C(X)$ by

$$(sf)(x) = f(s^{-1}x),$$

which in turn induces an action on $\mathcal{P}(X)$ (the space of regular Borel probability measures on X) by

$$\int_X f \, d(s\mu) = \int_X s^{-1}f \, d\mu.$$

Furthermore, restricting the latter action to the Dirac measures recovers the initial action.

Definition

Let X be a compact Γ-space. The Γ-action on X is called:

- *minimal* iff $\overline{\Gamma x} = X$ for all $x \in X$.
- *proximal* iff for every pair of points $x, y \in X$ there exists a net (s_i) in Γ such that $\lim s_i x = \lim s_i y$.
- *strongly proximal* iff the induced action on $\mathcal{P}(X)$ is proximal iff for every $\mu \in \mathcal{P}(X)$ there exists a Dirac measure in $\overline{\Gamma\mu}^{w^*}$.

If the Γ-action on X is both minimal and strongly proximal, then X is called a *(topological)* Γ*-boundary*.

The Furstenberg Boundary

Theorem (Furstenberg '73)

There exists a unique Γ-boundary which is universal in the sense that every other Γ-boundary is a continuous Γ-equivariant image of it. It is called the *Furstenberg boundary* of Γ and it is denoted by $\partial_F \Gamma$.

Proposition (Hamana '78, Kalantar-Kennedy '14)

The Hamana boundary and the Furstenberg boundary of Γ can be identified.

We will henceforth call it the FH-boundary of Γ and use the unifying $\partial_{FH}\Gamma$ to denote it.

Properties

- $\partial_{FH}\Gamma$ is extremally disconnected (i.e. the closure of any open subset is open).
- The point stabilizers of $\partial_{FH}\Gamma$ are amenable.
- The kernel of the Γ-action on $\partial_{FH}\Gamma$ is the amenable radical, $R_a(\Gamma)$, of Γ.
- Every Γ-map with $C(\partial_{FH}\Gamma)$ as its domain is automatically completely isometric.

The Reduced C^*-Algebra

Definition

The unitary representation $\lambda : \Gamma \to B(\ell^2(\Gamma))$ defined by

$$\lambda_s \delta_t = \delta_{st}$$

is called the *left regular representation* of Γ.

Definition

We define the *reduced* C^**-algebra* $C^*_r(\Gamma)$ of Γ to be the C^*-algebra generated by $\{\lambda_s : s \in \Gamma\}$ inside $B(\ell^2(\Gamma))$.

The reduced C^*-algebra comes equipped with the *canonical trace*, i.e. the map $\langle (.)\delta_e, \delta_e \rangle$, which is faithful.

The Reduced Crossed Product

Let $\mathcal{A}$ be a C^*-algebra on which Γ acts by *-isomorphisms and consider a faithful representation $\pi : \mathcal{A} \to B(\mathcal{H})$ of $\mathcal{A}$. We construct a new representation $\tilde{\pi} : \mathcal{A} \to B(\mathcal{H} \otimes \ell^2(\Gamma))$ by letting

$$\tilde{\pi}(a)(\xi \otimes \delta_t) = \pi(t^{-1}a)\xi \otimes \delta_t$$

for all $a \in \mathcal{A}$, $\xi \in \mathcal{H}$ and $t \in \Gamma$. Importantly, $\tilde{\pi}$ satisfies the covariance relation

$$\tilde{\pi}(sa) = (1_{\mathcal{H}} \otimes \lambda)(s)\tilde{\pi}(a)(1_{\mathcal{H}} \otimes \lambda)(s^{-1})$$

for all $s \in \Gamma$ and $a \in \mathcal{A}$.

Definition

We define the *reduced crossed product* $\mathcal{A} \rtimes_r \Gamma$ to be the C^*-algebra generated by $\tilde{\pi}(\mathcal{A}) \cup (1_{\mathcal{H}} \otimes \lambda)(\Gamma)$ inside $B(\mathcal{H} \otimes \ell^2(\Gamma))$.

The reduced crossed product comes equipped with a faithful *canonical conditional expectation* $E : \mathcal{A} \rtimes_r \Gamma \to \mathcal{A}$.

Let $(\pi, \mathcal{H}), (\sigma, \mathcal{K})$ be two unitary representations of Γ. We say that π is *weakly contained* in σ (denoted as $\pi \prec \sigma$) iff for any $\xi \in \mathcal{H}$, $F \subseteq \Gamma$ finite, and $\varepsilon > 0$, there exist $\eta_1, \eta_2, \ldots, \eta_n \in \mathcal{K}$ such that

$$\sup_{s \in F} \left| \langle \pi(s)\xi, \xi \rangle - \sum_{k=1}^{n} \langle \sigma(s)\eta_k, \eta_k \rangle \right| < \varepsilon.$$

We say that π and σ are *weakly equivalent* (denoted as $\pi \sim \sigma$) iff $\pi \prec \sigma$ and $\sigma \prec \pi$.

Definition

Γ is called C^*-simple iff $C^*_r(\Gamma)$ is simple iff the conditions $\pi \prec \lambda$ and $\pi \sim \lambda$ are equivalent for all unitary representations π of Γ.

The Main Theorem

Theorem (Kalantar-Kennedy '14, Breuillard-Kalantar-Kennedy-Ozawa '14-'16)

The following are equivalent:

1. Γ is C^*-simple.
2. $C(\partial_{FH}\Gamma) \rtimes_r \Gamma$ is simple.
3. $C(X) \rtimes_r \Gamma$ is simple for some Γ-boundary X.
4. The Γ-action on $\partial_{FH}\Gamma$ is (topologically) free.
5. The Γ-action on some Γ-boundary X is (topologically) free.

Reminder

A Γ-action on a space X is called *topologically free* iff the fixed point sets X^s have empty interior for all $s \in \Gamma \setminus \{e\}$.

Sketch of proof:

- Frolík's theorem implies that for $\partial_{FH}\Gamma$, which is extremally disconnected, topological freeness is equivalent to freeness (hence the parentheses).
- By universality and BCT, topological freeness of the Γ-action on some Γ-boundary X implies (topological) freeness of the Γ-action on $\partial_{FH}\Gamma$.
- Using the properties of the Furstenberg boundary, we show that if the Γ-action on some Γ-boundary X is not topologically free, then λ is not weakly contained in $\lambda_{\Gamma/\Gamma_x}$, for any $x \in X$. Since the point stabilizers of $\partial_{FH}\Gamma$ are amenable, we get that C^*-simplicity implies topological freeness.

Definition

Let $\Lambda \leqslant \Gamma$ be a subgroup. We define the *(left) quasi-regular* representation $\lambda_{\Gamma/\Lambda} : \Gamma \to B(\ell^2(\Gamma/\Lambda))$ of Γ associated to Λ by

$$\lambda_{\Gamma/\Lambda}(s)\delta_{t\Lambda} = \delta_{st\Lambda}$$

for all $s, t \in \Gamma$.

- Using the properties of the Hamana boundary and assuming freeness, we show that any unital *-representation of $C^*_r(\Gamma)$ extends to a faithful u.c.p. map on $C(\partial_{FH}\Gamma) \rtimes_r \Gamma$, and is thus injective, obtaining C^*-simplicity.
- Finally, the equivalence between freeness and simplicity of the associated reduced crossed products is obtained using the work of Kawamura-Tomiyama and Archbold-Spielberg from the early 90's.

Dixmier-Type Properties and C^*-Simplicity

Theorem (Haagerup '15-'16, Kennedy '15-'18)

Let τ_0 denote the canonical trace on $C^*_r(\Gamma)$. The following are equivalent:

1. Γ is C^*-simple.
2. $\tau_0 \in \overline{\{s\varphi : s \in \Gamma\}}^{w^*}$, for every state φ on $C^*_r(\Gamma)$.
3. $\tau_0 \in \overline{\text{conv}}^{w^*}\{s\varphi : s \in \Gamma\}$, for every state φ on $C^*_r(\Gamma)$.
4. $\omega(1)\tau_0 \in \overline{\text{conv}}^{w^*}\{s\omega : s \in \Gamma\}$, for every bounded linear functional ω on $C^*_r(\Gamma)$.
5. For all $t_1, t_2, \dots, t_m \in \Gamma \setminus \{e\}$,

$$0 \in \overline{\text{conv}}\{\lambda_s(\lambda_{t_1} + \lambda_{t_2} + \cdots + \lambda_{t_m})\lambda_s^* : s \in \Gamma\}.$$

Theorem (Haagerup '15, Kennedy '15-'18) [cont.]

6. For all $t_1, t_2, \ldots, t_m \in \Gamma \setminus \{e\}$ and all $\varepsilon > 0$, there exist $s_1, s_2, \ldots, s_n \in \Gamma$ such that
$$\left\| \sum_{k=1}^{n} \frac{1}{n} \lambda_{s_k t_j s_k^{-1}} \right\| < \varepsilon$$
for $j = 1, 2, \ldots, m$.

The last condition is historically important, as it is (essentially) Powers' averaging property.

Intrinsic Characterisation of C^*-Simplicity

Let $\mathrm{sub}(\Gamma)$ be the space of subgroups of Γ, equipped with the topology it inherits from the product topology on $\{0,1\}^{\Gamma}$. This is a compact space upon which Γ acts by conjugation. Consider also the subspace $\mathrm{sub}_a(\Gamma)$ of amenable subgroups of Γ.

Proposition

$\mathrm{sub}_a(\Gamma)$ is a compact Γ-subspace of $\mathrm{sub}(\Gamma)$.

Definition

A compact Γ-subspace $X \subseteq \mathrm{sub}(\Gamma)$ is called a *uniformly recurrent subgroup (URS)* iff it is minimal. Such an X is *trivial* iff $X = \{\{e\}\}$, and *amenable* iff $X \subseteq \mathrm{sub}_a(\Gamma)$.

Intrinsic Characterisation of C^*-Simplicity

Theorem (Kennedy '15-'18)

Γ is C^*-simple iff it has no non-trivial amenable uniformly recurrent subgroups.

In reality, C^*-simplicity turns out to be equivalent to the triviality of the *Furstenberg URS*, i.e. $X = \{\Gamma_x : x \in \partial_{FH}\Gamma\}$, which we know is amenable.

The Unique Trace Property

We say that Γ has the *unique trace property* iff $C^*_r(\Gamma)$ admits no traces other than the canonical one.

Theorem (Breuillard-Kalantar-Kennedy-Ozawa '14-'16)

Let $s \in \Gamma$. Then $\tau(\lambda_s) = 0$ for all tracial states τ on $C^*_r(\Gamma)$ iff $s \notin R_a(\Gamma)$. In particular, Γ has the unique trace property iff $R_a(\Gamma)$ is trivial.

Corollary

The following are equivalent:

1. Γ has the unique trace property.
2. The Γ-action on $\partial_{FH}\Gamma$ is faithful.
3. The Γ-action on some Γ-boundary X is faithful.

As in the case of C^*-simplicity, Haagerup also formulated a Dixmier-type characterisation of the unique trace property.

Theorem (Haagerup '15-'16)

Let $t \in \Gamma$. Then $t \notin R_a(\Gamma)$ iff

$$0 \in \overline{\text{conv}}\{\lambda_{sts^{-1}} : s \in \Gamma\}. \tag{1}$$

In particular, Γ has the unique trace property iff (1) holds for all $t \in \Gamma \setminus e$.

C^*-Simplicity vs the UTP

It is clear from all the characterisations we have given that C^*-simplicity implies the unique trace property, but it is not immediate that the former is strictly stronger than the latter. However, Le Boudec and Matte Bon managed to find examples of non-C^*-simple groups which have trivial amenable radical, in the realm of geometric group theory.

Exactness

Definition

A C^*-algebra $\mathcal{A}$ is called *exact* iff the functor $(\mathcal{A} \otimes -)$ is exact (in the categorical sense).

Definition

Γ is called *exact* iff $C_r^*(\Gamma)$ is exact as a C^*-algebra.

Amenable Actions

Definition

A compact Γ-space X is called *amenable* iff there exists a net of continuous maps $m_i : X \to \mathcal{P}(\Gamma)$ such that

$$\lim_i \big(\sup_{x \in X} \|s m_i^x - m_i^{sx}\|_1 \big) = 0$$

for all $s \in \Gamma$. Such a net is called an *approximate invariant continuous mean (a.i.c.m.)*. The Γ-action on such a space is also called *amenable*.

Definition

Γ is called *amenable at infinity* iff it acts amenably on some compact Γ-space.

Dynamical Characterisations of Exactness

Theorem (Ozawa '00, Anantharaman-Delaroche '02)

Γ is exact iff it the left translation action on $\beta\Gamma$ is amenable iff it is amenable at infinity.

Theorem (Kalantar-Kennedy '14)

Γ is exact iff the Γ-action on $\partial_{FH}\Gamma$ is amenable.

Thank you!