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Tensor algebras of C^*-correspondences and their C^*-envelopes

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Abstract

We show that the C^*-envelope of the tensor algebra of an arbitrary C^*-correspondence $\mathcal{X}$ coincides with the Cuntz–Pimsner algebra $\mathcal{O}_{\mathcal{X}}$, as defined by Katsura [T. Katsura, On C^*-algebras associated with C^*-correspondences, J. Funct. Anal. 217 (2004) 366–401]. This improves earlier results of Muhly and Solel [P.S. Muhly, B. Solel, Tensor algebras over C^*-correspondences: Representations, dilations and C^*-envelopes, J. Funct. Anal. 158 (1998) 389–457] and Fowler, Muhly and Raeburn [N. Fowler, P. Muhly, I. Raeburn, Representations of Cuntz–Pimsner algebras, Indiana Univ. Math. J. 52 (2003) 569–605], who came to the same conclusion under the additional hypothesis that $\mathcal{X}$ is strict and faithful.
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1. Introduction

Fowler, Muhly and Raeburn have recently characterized [5, Theorem 5.3] the C^*-envelope of the tensor algebra $\mathcal{T}_{\mathcal{X}}^+$ of a *faithful and strict* C^*-correspondence $\mathcal{X}$, as the associated universal Cuntz–Pimsner algebra. Their proof is based on a gauge invariant uniqueness theorem and earlier elaborate results of Muhly and Solel [11]. Beyond faithful strict C^*-correspondences, little is

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known: if $\mathcal{X}$ is strict, but not necessary faithful, then the C^*-envelope of $\mathcal{T}^+_{\mathcal{X}}$ is known to be a quotient of the associated Toeplitz–Cuntz–Pimsner algebra, without any further information [11, Theorem 6.4]. In [5, Remark 5.4], the authors ask whether the above mentioned conditions on $\mathcal{X}$ are necessary for the validity of their [5, Theorem 5.3].

In this note we answer the question of Fowler, Muhly and Raeburn [5] (and Muhly and Solel [11]) by showing that the C^*-envelope of the tensor algebra of an *arbitrary* C^*-correspondence $\mathcal{X}$ coincides with the Cuntz–Pimsner algebra $\mathcal{O}_{\mathcal{X}}$, as defined by Katsura in [8]. Our proof does not require any of the results from [11] and is modelled upon the proof of our recent result [7] that identifies the C^*-envelope of the tensor algebra of a directed graph. We also make use of the result of Muhly and Tomforde [12] that generalizes the process of adding tails to a graph to the context of C^*-correspondences.

2. Preliminaries

Let $\mathcal{A}$ be a C^*-algebra and $\mathcal{X}$ be a (right) Hilbert $\mathcal{A}$-module, whose inner product is denoted as $\langle\cdot\,|\,\cdot\rangle$. Let $\mathcal{L}(\mathcal{X})$ be the adjointable operators on $\mathcal{X}$ and let $\mathcal{K}(\mathcal{X})$ be the norm-closed subalgebra of $\mathcal{L}(\mathcal{X})$ generated by the operators $\theta_{\xi,\eta}$, $\xi,\eta\in\mathcal{X}$, where $\theta_{\xi,\eta}(\zeta)=\xi\langle\eta|\zeta\rangle$, $\zeta\in\mathcal{X}$.

A Hilbert $\mathcal{A}$-module $\mathcal{X}$ is said to be a *C^*-correspondence* over $\mathcal{A}$ provided that there exists a $*$-homomorphism $\phi_{\mathcal{X}}:\mathcal{A}\to\mathcal{L}(\mathcal{X})$. We refer to $\phi_{\mathcal{X}}$ as the left action of $\mathcal{A}$ on $\mathcal{X}$. A C^*-correspondence $\mathcal{X}$ over $\mathcal{A}$ is said to be *faithful* if and only if the map $\phi_{\mathcal{X}}$ is faithful. A C^*-correspondence $\mathcal{X}$ over $\mathcal{A}$ is called *strict* iff $\overline{[\phi_{\mathcal{X}}(\mathcal{A})\mathcal{X}]}\subseteq\mathcal{X}$ is complemented, as a submodule of the Hilbert $\mathcal{A}$-module $\mathcal{X}$. In particular, if $\overline{[\phi_{\mathcal{X}}(\mathcal{A})\mathcal{X}]}=\mathcal{X}$, i.e., the map $\phi_{\mathcal{X}}$ is non-degenerate, then $\mathcal{X}$ is said to be *essential*.

From a given C^*-correspondence $\mathcal{X}$ over $\mathcal{A}$, one can form new C^*-correspondences over $\mathcal{A}$, such as the *n-fold ampliation* or direct sum $\mathcal{X}^{(n)}$ [9, p. 5] and the *n-fold interior tensor product* $\mathcal{X}^{\otimes n}\equiv\mathcal{X}\otimes_{\phi_{\mathcal{X}}}\mathcal{X}\otimes_{\phi_{\mathcal{X}}}\cdots\otimes_{\phi_{\mathcal{X}}}\mathcal{X}$ [9, p. 39], $n\in\mathbb{N}$ ($\mathcal{X}^{\otimes 0}\equiv\mathcal{A}$). These operation are defined within the category of C^*-correspondences over $\mathcal{A}$. (See [9] for more details.)

A *representation* (π,t) of a C^*-correspondence $\mathcal{X}$ over $\mathcal{A}$ on a C^*-algebra $\mathcal{B}$ consists of a $*$-homomorphism $\pi:\mathcal{A}\to\mathcal{B}$ and a linear map $t:\mathcal{X}\to\mathcal{B}$ so that

$$\text{(i)}\quad t(\xi)^*t(\eta)=\pi\big(\langle\xi|\eta\rangle\big),\quad \text{for } \xi,\eta\in\mathcal{X},$$
$$\text{(ii)}\quad \pi(a)t(\xi)=t\big(\phi_{\mathcal{X}}(a)\xi\big),\quad \text{for } a\in\mathcal{A},\ \xi\in\mathcal{X}.$$

For a representation (π,t) of a C^*-correspondence $\mathcal{X}$ there exists a $*$-homomorphism $\psi_t:\mathcal{K}(\mathcal{X})\to\mathcal{B}$ so that $\psi_t(\theta_{\xi,\eta})=t(\xi)t(\eta)^*$, for $\xi,\eta\in\mathcal{X}$. Following Katsura [8], we say that the representation (π,t) is *covariant* iff $\psi_t(\phi_{\mathcal{X}}(a))=\pi(a)$, for all $a\in\mathcal{J}_{\mathcal{X}}$, where

$$\mathcal{J}_{\mathcal{X}}\equiv\phi_{\mathcal{X}}^{-1}\big(\mathcal{K}(\mathcal{X})\big)\cap(\ker\phi_{\mathcal{X}})^{\perp}.$$

If (π,t) is a representation of $\mathcal{X}$ then the C^*-algebra (respectively norm-closed algebra) generated by the images of π and t is denoted as $C^*(\pi,t)$ (respectively $\mathrm{alg}(\pi,t)$). There is a universal representation $(\bar\pi_{\mathcal{A}},\bar t_{\mathcal{X}})$ for $\mathcal{X}$ and the C^*-algebra $C^*(\bar\pi_{\mathcal{A}},\bar t_{\mathcal{X}})$ is the Toeplitz–Cuntz–Pimsner algebra $\mathcal{T}_{\mathcal{X}}$. Similarly, the Cuntz–Pimsner algebra $\mathcal{O}_{\mathcal{X}}$ is the C^*-algebra generated by the image of the universal covariant representation $(\pi_{\mathcal{A}},t_{\mathcal{X}})$ for $\mathcal{X}$.

A concrete presentation of both $\mathcal{T}_{\mathcal{X}}$ and $\mathcal{O}_{\mathcal{X}}$ can be given in terms of the generalized Fock space $\mathcal{F}_{\mathcal{X}}$ which we now describe. The *Fock space* $\mathcal{F}_{\mathcal{X}}$ over the correspondence $\mathcal{X}$ is defined to be the direct sum of the $\mathcal{X}^{\otimes n}$ with the structure of a direct sum of C^*-correspondences over $\mathcal{A}$,

$$\mathcal{F}_{\mathcal{X}} = \mathcal{A} \oplus \mathcal{X} \oplus \mathcal{X}^{\otimes 2} \oplus \cdots.$$

Given $\xi \in \mathcal{X}$, the (left) creation operator $t_\infty(\xi) \in \mathcal{L}(\mathcal{F}_{\mathcal{X}})$ is defined by the formula

$$t_\infty(\xi)(a, \zeta_1, \zeta_2, \ldots) = (0, \xi a, \xi \otimes \zeta_1, \xi \otimes \zeta_2, \ldots),$$

where $\zeta_n \in \mathcal{X}^{\otimes n}$, $n \in \mathbb{N}$. Also, for $a \in \mathcal{A}$, we define $\pi_\infty(a) \in \mathcal{L}(\mathcal{F}_{\mathcal{X}})$ to be the diagonal operator with $\phi_{\mathcal{X}}(a) \otimes id_{n-1}$ at its $\mathcal{X}^{\otimes n}$th entry. It is easy to verify that (π_∞, t_∞) is a representation of $\mathcal{X}$ which is called the *Fock representation* of $\mathcal{X}$. Fowler and Raeburn [4] (respectively Katsura [8]) have shown that the C^*-algebra $C^*(\pi_\infty, t_\infty)$ (respectively $C^*(\pi_\infty, t_\infty)/\mathcal{K}(\mathcal{F}_{\mathcal{X}}\mathcal{J}_{\mathcal{X}})$) is isomorphic to $\mathcal{T}_{\mathcal{X}}$ (respectively $\mathcal{O}_{\mathcal{X}}$).

Definition 2.1. The *tensor algebra* of a C^*-correspondence $\mathcal{X}$ over $\mathcal{A}$ is the norm-closed algebra $\mathrm{alg}(\bar{\pi}_{\mathcal{A}}, \bar{t}_{\mathcal{X}})$ and is denoted as $\mathcal{T}^+_{\mathcal{X}}$.

According to [4,8], the algebras $\mathcal{T}^+_{\mathcal{X}} \equiv \mathrm{alg}(\bar{\pi}_{\mathcal{A}}, \bar{t}_{\mathcal{X}})$ and $\mathrm{alg}(\pi_\infty, t_\infty)$ are completely isometrically isomorphic and we will therefore identify them. The main result of this paper implies that $\mathcal{T}^+_{\mathcal{X}}$ is also completely isometrically isomorphic to $\mathrm{alg}(\pi_{\mathcal{A}}, t_{\mathcal{X}})$.

3. Main result

We begin with a useful description of the norm in $\mathcal{X}^{(n)}$.

Lemma 3.1. *Let $\mathcal{X}$, $\mathcal{Y}$ be Hilbert $\mathcal{A}$-modules and let $\phi : \mathcal{A} \to \mathcal{L}(\mathcal{Y})$ be an injective $*$-homomorphism. If $(\xi_i)_{i=1}^n \in \mathcal{X}^{(n)}$, then*

$$\left\|(\xi_i)_{i=1}^n\right\| = \sup\left\{\left\|(\xi_i \otimes_\phi u)_{i=1}^n\right\| \;\middle|\; u \in \mathcal{Y},\ \|u\| = 1\right\}. \tag{1}$$

Proof. Let us denote by M the supremum in (1). Then, using the fact that ϕ is injective and therefore isometric,

$$\begin{aligned} M^2 &= \sup\left\{\left\|\sum_{i=1}^n \langle u | \phi(\langle \xi_i | \xi_i \rangle) u \rangle\right\| \;\middle|\; u \in \mathcal{Y}, \|u\| = 1\right\} \\ &= \sup\left\{\left\|\left(\phi\left(\langle \xi_i | \xi_i \rangle^{1/2}\right) u\right)_i\right\|^2 \;\middle|\; u \in \mathcal{Y},\ \|u\| = 1\right\} \\ &= \left\|\begin{pmatrix} 0 & 0 & \ldots & \phi(\langle \xi_1 | \xi_1 \rangle^{1/2}) \\ 0 & 0 & \ldots & \phi(\langle \xi_2 | \xi_2 \rangle^{1/2}) \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \phi(\langle \xi_n | \xi_n \rangle^{1/2}) \end{pmatrix}\right\|^2 = \left\|\phi\left(\sum_{i=1}^n \langle \xi_i | \xi_i \rangle\right)\right\| = \left\|(\xi)_i\right\|^2 \end{aligned}$$

and the conclusion follows. □

In the proof of our next lemma we make use of the right creation operators. If $\mathcal{Y}$ is a C^*-correspondence over $\mathcal{A}$ and $\xi \in \mathcal{Y}^{\otimes k}$, then define the right creation operator R_ξ by the formula

$$R_\xi(a, \zeta_1, \zeta_2, \ldots) = \big(\underbrace{0, 0, \ldots, 0}_{k}, \big(\phi_{\mathcal{X}}(a) \otimes id_{k-1}\big)(\xi), \zeta_1 \otimes \xi, \zeta_2 \otimes \xi, \ldots\big),$$

$\zeta_n \in \mathcal{Y}^{\otimes n}$, $n \in \mathbb{N}$. The operator, R_ξ may not be adjointable but it is nevertheless bounded by $\|\xi\|$ and commutes with $\mathrm{alg}(\pi_\infty, t_\infty)$.

Lemma 3.2. *If $\mathcal{X}$ be a faithful C^*-correspondence over $\mathcal{A}$, then*

$$\|A\| = \inf\big\{\|A + K\| \mid K \in M_n\big(\mathcal{K}(\mathcal{F}_{\mathcal{X}})\big)\big\}$$

for all $A \in M_n(\mathcal{T}_{\mathcal{X}}^+)$, $n \in \mathbb{N}$.

Proof. Let $K \in M_n(\mathcal{K}(\mathcal{F}_{\mathcal{X}}))$ be an $n \times n$ matrix with entries in $\mathcal{K}(\mathcal{F}_{\mathcal{X}})$ and let $\epsilon > 0$. We choose unit vector $\xi \in \mathcal{F}_{\mathcal{X}}^{(n)}$ so that $\|A\xi\| \geqslant \|A\| - \epsilon$. Since $K \in M_n(\mathcal{K}(\mathcal{F}_{\mathcal{X}}))$, there exists $k \in \mathbb{N}$ so that $\|K R_u^{(n)}\| \leqslant \epsilon$, for all unit vectors $u \in \mathcal{X}^{\otimes k}$. (Here $R_u^{(n)}$ denotes the nth ampliation of the right creation operator R_u.) Note that for any vector $u \in \mathcal{X}^{\otimes k}$ we have

$$\big\|R_u^{(n)} A\xi\big\| = \|A\xi \otimes u\|.$$

Therefore, using Lemma 3.1, we choose unit vector $u \in \mathcal{X}^{\otimes k}$ so that

$$\big\|R_u^{(n)} A\xi\big\| \geqslant \|A\xi\| - \epsilon \geqslant \|A\| - 2\epsilon.$$

We compute,

$$\begin{aligned}\|A + K\| &\geqslant \big\|(A + K) R_u^{(n)}\xi\big\| \geqslant \big\|A R_u^{(n)}\xi\big\| - \epsilon = \big\|R_u^{(n)} A\xi\big\| - \epsilon \\ &\geqslant \|A\| - 3\epsilon.\end{aligned}$$

Since ϵ and K are arbitrary, the proof is complete. □

Corollary 3.3. *Let $\mathcal{X}$ be a faithful C^*-correspondence over $\mathcal{A}$, and let $(\pi_{\mathcal{A}}, t_{\mathcal{X}})$ be the universal covariant representation of $\mathcal{X}$. Then, there exists a complete isometry*

$$\tau_{\mathcal{X}} : \mathcal{T}_{\mathcal{X}}^+ \to \mathrm{alg}(\pi_{\mathcal{A}}, t_{\mathcal{X}})$$

so that $\tau_{\mathcal{X}}(\pi_\infty(a)) = \pi_{\mathcal{A}}(a)$, for all $a \in \mathcal{A}$, and $\tau_{\mathcal{X}}(t_\infty(\xi)) = t_{\mathcal{X}}(\xi)$, for all $\xi \in \mathcal{X}$.

In particular, the algebra $\mathrm{alg}(\pi_{\mathcal{A}}, t_{\mathcal{X}})$ is completely isometrically isomorphic to the tensor algebra $\mathcal{T}_{\mathcal{X}}^+$.

Proof. Let $\tau_{\mathcal{X}}$ be the restriction of the natural quotient map

$$C^*(\pi_\infty, t_\infty) \to C^*(\pi_\infty, t_\infty)/\mathcal{K}(\mathcal{F}_{\mathcal{X}} \mathcal{J}_{\mathcal{X}})$$

on the non-selfadjoint subalgebra $\mathrm{alg}(\pi_\infty, t_\infty)$. By Lemma 3.2, this map is a complete isometry. $\square$

Remark 3.4. Note that the above lemma already implies the result of Fowler, Muhly and Raeburn [5, Theorem 5.3] without their requirement of $\mathcal{X}$ being strict.

We now remove the requirement of $\mathcal{X}$ being faithful from the statement of the above lemma. In the special case of a graph correspondence, this was done in [7] with the help of a well-known process called "adding tails to a graph." This process has been generalized to arbitrary correspondences by Muhly and Tomforde [12]. Indeed, let $\mathcal{X}$ be an arbitrary C^*-correspondence over $\mathcal{A}$ and let $\mathfrak{T} \equiv c_0(\ker \phi_{\mathcal{X}})$ consist of all null sequences in $\ker \phi_{\mathcal{X}}$. Muhly and Tomforde show that there exists a well-defined left action of $\mathcal{B} \equiv \mathcal{A} \oplus \mathfrak{T}$ on $\mathcal{Y} \equiv \mathcal{X} \oplus \mathfrak{T}$ so that $\mathcal{Y}$ becomes a *faithful* C^*-correspondence over $\mathcal{B}$. One can view $\mathcal{A}$ and the C^*-correspondence $\mathcal{X}$ as a subsets of $\mathcal{B}$ and $\mathcal{Y}$ respectively, via the identifications

$$\mathcal{A} \ni a \to (a, 0) \in \mathcal{A} \oplus 0,$$

$$\mathcal{X} \ni \xi \to (\xi, 0) \in \mathcal{X} \oplus 0$$

and by noting that the action of $\phi_{\mathcal{Y}}$ on $\mathcal{A} \oplus 0$ coincides with that of $\phi_{\mathcal{X}}$ on $\mathcal{A}$. (The restriction of a representation (π, t) of $\mathcal{Y}$ on that subset of $\mathcal{Y}$ will be denoted as $(\pi_{|\mathcal{A}}, t_{|\mathcal{X}})$ and is indeed a representation of $\mathcal{X}$.) In [12, Theorem 4.3(b)] it is shown that if (π, t) is a covariant representation of $\mathcal{Y}$, then $(\pi_{|\mathcal{A}}, t_{|\mathcal{X}})$ is a covariant representation of $\mathcal{X}$.

Lemma 3.5. *Let $\mathcal{X}$ be a C^*-correspondence over $\mathcal{A}$, and let $(\pi_{\mathcal{A}}, t_{\mathcal{X}})$ be the universal covariant representation of $\mathcal{X}$. Then, there exists a complete isometry*

$$\tau_{\mathcal{X}} : \mathcal{T}_{\mathcal{X}}^+ \to \mathrm{alg}(\pi_{\mathcal{A}}, t_{\mathcal{X}})$$

so that $\tau_{\mathcal{X}}(\pi_\infty(a)) = \pi_{\mathcal{A}}(a)$, for all $a \in \mathcal{A}$, and $\tau_{\mathcal{X}}(t_\infty(\xi)) = t_{\mathcal{X}}(\xi)$, for all $\xi \in \mathcal{X}$.

Proof. Let (π_∞, t_∞) be the Fock representation of $\mathcal{Y}$ and note that [8, Corollary 4.5] shows that

$$\pi_\infty(\mathcal{B}) \cap \psi_{t_\infty}\big(\mathcal{K}(\mathcal{Y})\big) = \{0\}.$$

Therefore, the restriction $(\pi_{\infty|\mathcal{A}}, t_{\infty|\mathcal{X}})$ satisfies the same property and so [8, Theorem 6.2] implies that the integrated representation $\pi_{\infty|\mathcal{A}} \times t_{\infty|\mathcal{X}}$ is a C^*-isomorphism from the universal Toeplitz algebra $\mathcal{T}_{\mathcal{X}}$ onto $C^*(\pi_{\infty|\mathcal{X}}, t_{\infty|\mathcal{X}})$. We therefore view $\mathcal{T}_{\mathcal{X}}^+$ as a subalgebra of $\mathcal{T}_{\mathcal{Y}}^+$.

Corollary 3.3 shows now that there exists a complete isometry

$$\tau_{\mathcal{Y}} : \mathcal{T}_{\mathcal{Y}}^+ \to \mathrm{alg}(\pi_{\mathcal{B}}, t_{\mathcal{Y}})$$

so that $\tau_{\mathcal{Y}}(\pi_\infty(b)) = \pi_{\mathcal{B}}(b)$, for all $b \in \mathcal{B}$, and $\tau_{\mathcal{Y}}(\pi_\infty(\xi)) = t_{\mathcal{Y}}(\xi)$, for all $\xi \in \mathcal{Y}$. As we discussed earlier, [12, Theorem 4.3(b)] shows that the restriction $(\pi_{\mathcal{B}|\mathcal{A}}, t_{\mathcal{Y}|\mathcal{X}})$ is covariant for $\mathcal{X}$. Since it is also injective, the gauge invariant uniqueness theorem [8, Theorem 6.4] shows that the restriction $\tau_{\mathcal{X}} \equiv \tau_{\mathcal{Y}}|_{\mathcal{T}_{\mathcal{X}}}$ has range isomorphic to $\mathrm{alg}(\pi_{\mathcal{A}}, t_{\mathcal{X}})$ and satisfies the desired properties. $\square$

Let $\mathcal{B}$ be a C^*-algebra and let $\mathcal{B}^+$ be a (nonselfadjoint) subalgebra of $\mathcal{B}$ which generates $\mathcal{B}$ as a C^*-algebra and contains a two-sided contractive approximate unit for $\mathcal{B}$, i.e., $\mathcal{B}^+$ is an essential subalgebra for $\mathcal{B}$. A two-sided ideal $\mathcal{J}$ of $\mathcal{B}^+$ is said to be a *boundary ideal* for $\mathcal{B}^+$ if and only if the quotient map $\pi : \mathcal{B} \to \mathcal{B}/\mathcal{J}$ is a complete isometry when restricted to $\mathcal{B}^+$. It is a result of Hamana [6], following the seminal work of Arveson [1], that there exists a boundary ideal $\mathcal{J}_S(\mathcal{B}^+)$, the *Shilov boundary ideal*, that contains all other boundary ideals. In that case, the quotient $\mathcal{B}/\mathcal{J}_S(\mathcal{B}^+)$ is called the C^**-envelope* of $\mathcal{B}^+$ and it is denoted as $\mathrm{C}^*_{\mathrm{env}}(\mathcal{B}^+)$. The C^*-envelope is unique in the following sense. Assume that $\phi' : \mathcal{B}^+ \to \mathcal{B}'$ is a completely isometric isomorphism of $\mathcal{B}^+$ onto an essential subalgebra of a C^*-algebra $\mathcal{B}'$ and suppose that the Shilov boundary for $\phi'(\mathcal{B}^+) \subseteq \mathcal{B}'$ is zero. Then $\mathcal{B}$ and $\mathcal{B}'$ are $*$-isomorphic, via an isomorphism ϕ so that $\phi(\pi(x)) = \phi'(x)$, for all $x \in \mathcal{B}$.

In the case where an operator algebra $\mathcal{B}^+$ has no contractive approximate identity, the $\mathrm{C}^*_{\mathrm{env}}(\mathcal{B}^+)$ is defined by utilizing the unitization [10] $(\mathcal{B}^+)_1$ of $\mathcal{B}^+$: the C^*-envelope of $\mathcal{B}^+$ is the C^*-subalgebra of $\mathrm{C}^*_{\mathrm{env}}((\mathcal{B}^+)_1)$ generated by $\mathcal{B}^+$. (See [2,3] for a comprehensive discussion regarding the implications of [10] on the theory of C^*-envelopes.)

Lemma 3.6. *Let $\mathcal{B}$ be a non-unital C^*-algebra and let $\mathcal{J} \subseteq \mathcal{B}_1$ be a closed two-sided ideal in its unitization. If $\mathcal{J} \cap \mathcal{B} = \{0\}$ then $\mathcal{J} = \{0\}$.*

Proof. Assume that $\mathcal{J} \neq \{0\}$. Since $\mathcal{B}_1 \subseteq \mathcal{B}$ has codimension 1, $\mathcal{J}$ is of the form $\mathcal{J} = [\{A + \lambda I\}]$, for some $A \in \mathcal{B}$ and non-zero $\lambda \in \mathbb{C}$. Then, easy manipulations show that there is no loss of generality assuming that $\lambda \in \mathbb{R}$ (because $\mathcal{J}\mathcal{J}^* \neq 0$), A is selfadjoint (because $\mathcal{J} \cap \mathcal{J}^* \neq 0$) and

$$(A + \lambda I)^2 = A + \lambda, \tag{2}$$

after perhaps scaling (since $\mathcal{J}^2 \neq 0$). It is easy to see now that (2) implies that $A = -P$, for some projection $P \in \mathcal{B}$. But then, $(I - P)\mathcal{B} = 0$ and so P is a unit for $\mathcal{B}$, a contradiction. □

We have arrived to the main result of the paper.

Theorem 3.7. *If $\mathcal{X}$ is a C^*-correspondence over $\mathcal{A}$, then the C^*-envelope of $\mathcal{T}^+_{\mathcal{X}}$ coincides with the universal Cuntz–Pimsner algebra $\mathcal{O}_{\mathcal{X}}$.*

Proof. According to Lemma 3.5, it suffices to show that the C^*-envelope of $\mathrm{alg}(\pi_{\mathcal{A}}, t_{\mathcal{X}})$ equals $\mathcal{O}_{\mathcal{X}}$.

Assume first that $\mathrm{alg}(\pi_{\mathcal{A}}, t_{\mathcal{X}})$ is unital. In light of the above discussion, we need to verify that the Shilov boundary ideal $\mathcal{J}_S(\mathrm{alg}(\pi_{\mathcal{A}}, t_{\mathcal{X}}))$ is zero. However, the maximality of $\mathcal{J}_S(\mathrm{alg}(\pi_{\mathcal{A}}, t_{\mathcal{X}}))$ and the invariance of $\mathrm{alg}(\pi_{\mathcal{A}}, t_{\mathcal{X}})$ under the gauge action of $\mathbb{T}$ on $\mathcal{O}_{\mathcal{X}}$ imply that $\mathcal{J}_S(\mathrm{alg}(\pi_{\mathcal{A}}, t_{\mathcal{X}}))$ is a gauge-invariant ideal. By the gauge invariant uniqueness theorem [8, Theorem 6.4], any non-zero gauge-invariant ideal has non-zero intersection with $\pi_{\mathcal{A}}(\mathcal{A})$. Hence $\mathcal{J}_S(\mathrm{alg}(\pi_{\mathcal{A}}, t_{\mathcal{X}})) = \{0\}$, or otherwise the quotient map would not be faithful on $\mathrm{alg}(\pi_{\mathcal{A}}, t_{\mathcal{X}})$.

Assume now that $\mathrm{alg}(\pi_{\mathcal{A}}, t_{\mathcal{X}})$ is not unital. We distinguish two cases.

If $\mathcal{O}_{\mathcal{X}}$ has a unit $I \in \mathcal{O}_{\mathcal{X}}$ then let

$$\mathrm{alg}(\pi_{\mathcal{A}}, t_{\mathcal{X}})_1 \equiv \mathrm{alg}(\pi_{\mathcal{A}}, t_{\mathcal{X}}) + \mathbb{C}I \subseteq \mathcal{O}_{\mathcal{X}}.$$

Clearly, $\mathrm{alg}(\pi_{\mathcal{A}}, t_{\mathcal{X}})_1$ is gauge invariant and so a repetition of the arguments in the second paragraph of the proof shows that

$$\mathrm{C}^*_{\mathrm{env}}\big(\mathrm{alg}(\pi_{\mathcal{A}}, t_{\mathcal{X}})_1\big) = \mathcal{O}_{\mathcal{X}}.$$

The C^*-subalgebra of $\mathcal{O}_{\mathcal{X}}$ generated by $\mathrm{alg}(\pi_{\mathcal{A}}, t_{\mathcal{X}})$ equals $\mathcal{O}_{\mathcal{X}}$, which by convention will be its C^*-envelope.

Finally, if $\mathcal{O}_{\mathcal{X}}$ does not have a unit then unitize $\mathcal{O}_{\mathcal{X}}$ by joining a unit I and let

$$\mathrm{alg}(\pi_{\mathcal{A}}, t_{\mathcal{X}})_1 \equiv \mathrm{alg}(\pi_{\mathcal{A}}, t_{\mathcal{X}}) + \mathbb{C}I \subseteq \mathcal{O}_{\mathcal{X}} + \mathbb{C}I.$$

Since the Shilov ideal $\mathcal{J}_S(\mathrm{alg}(\pi_{\mathcal{A}}, t_{\mathcal{X}})_1)$ is gauge invariant,

$$\mathcal{J}_S\big(\mathrm{alg}(\pi_{\mathcal{A}}, t_{\mathcal{X}})_1\big) \cap \mathcal{O}_{\mathcal{X}} \subseteq \mathcal{O}_{\mathcal{X}}$$

is gauge invariant. Therefore,

$$\mathcal{J}_S\big(\mathrm{alg}(\pi_{\mathcal{A}}, t_{\mathcal{X}})_1\big) \cap \mathcal{O}_{\mathcal{X}} = \{0\},$$

or else it meets $\pi_{\mathcal{A}}(\mathcal{A})$. By Lemma 3.6, $\mathcal{J}_S(\mathrm{alg}(\pi_{\mathcal{A}}, t_{\mathcal{X}})_1) = \{0\}$ and so $\mathrm{C}^*_{\mathrm{env}}(\mathrm{alg}(\pi_{\mathcal{A}}, t_{\mathcal{X}})_1) = \mathcal{O}_{\mathcal{X}} + \mathbb{C}I$. The C^*-subalgebra of $\mathcal{O}_{\mathcal{X}} + \mathbb{C}I$ generated by $\mathrm{alg}(\pi_{\mathcal{A}}, t_{\mathcal{X}})$ is $\mathcal{O}_{\mathcal{X}}$, and the conclusion follows. □

Remark 3.8. In [5, p. 596], it is claimed that if a $\mathcal{X}$ is a C^*-correspondence over $\mathcal{A}$, with universal Toeplitz representation $(\bar{\pi}_{\mathcal{A}}, \bar{t}_{\mathcal{X}})$, then $\bar{\pi}_{\mathcal{A}}$ maps an approximate unit of $\mathcal{A}$ to an approximate unit for both $\mathcal{T}_{\mathcal{X}}$ and $\mathcal{T}^+_{\mathcal{X}}$. It is not hard to see that this claim is valid if and only if $\phi_{\mathcal{X}}$ is nondegenerate. Therefore, there is a gap in the proof of [5, Theorem 5.3] in the case where $\mathcal{X}$ is strict but not essential. Nevertheless, our Theorem 3.7 incorporates all possible cases and hence completes the proof of [5, Theorem 5.3].

We now obtain one of the main results of [7] as a corollary.

Corollary 3.9 *(*[7, Theorem 2.5]*). If G is a countable directed graph then the C^*-envelope of $\mathcal{T}_+(G)$ coincides with the universal Cuntz–Krieger algebra associated with G.*

Note that in [7], the proof of the above corollary is essentially self-contained and avoids the heavy machinery used in this paper. The reader would actually benefit from reading that proof and then making comparisons with the proof of Theorem 3.7 here.

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