ATHENS LECTURES

Subalgebras of graph C*-algebras

Summer Lectures on Operator Algebras

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Introduction

The purpose of these lectures is to present some interesting classes of (non-selfadjoint) operator algebras.

First, we introduce the non-commutative analogues of the (non-selfadjoint) disc algebra $\mathbb{A}(\mathbb{D})$. These are norm-closed unital operator algebras generated by n isometries $S_1, \ldots, S_n$ (resp. $L_1, \ldots, L_n$) satisfying $\sum_i S_i S_i^* = I$ (resp. $\sum_i L_i L_i^* < I$). The first algebra, $\mathcal{A}_n$, generates the Cuntz algebra $\mathcal{O}_n$, just as $\mathbb{A}(\mathbb{D})$ generates $C(\mathbb{T})$. The second one, $\mathbb{A}_n$, generates the Cuntz-Toeplitz algebra $\mathcal{T}\mathcal{O}_n$, just as $\mathbb{A}(\mathbb{D})$ generates the Toeplitz algebra.

In Parts I-III, we prove uniqueness of $\mathcal{O}_n$ and represent it explicitly on $L^2[0,1]$ (*the interval picture*) and on a graph (*the Cantorised interval picture*). We also present a first connection of gauge automorphisms with the notion of Cesaro summability.

In Part IV we use the notion of the C^*-envelope of a non-selfadjoint algebra and its Shilov ideal in order to identify the non-commutative disc algebra $\mathcal{A}_n \subseteq \mathcal{O}_n$ with a quotient of $\mathbb{A}_n \subseteq \mathcal{T}\mathcal{O}_n$.

These ideas are then generalized to the case of an algebra acting on a directed graph G. In Part V the gauge invariance uniqueness theorem is used to prove that $C_{env}(\mathcal{A}_G)$ and $C^*(G)$ are isomorphic as C^*-algebras, analogously to what was done in Parts III and IV. In Part VI we consider the case where G is a directed graph with two-coloured edges and 'freeness' is restricted by commutation relations.

PART I

The Cuntz Algebras, intuitively.

We will prove that any C^*-algebra generated by n isometries with orthogonal ranges summing to I is isometrically isomorphic to the Cuntz algebras $\mathcal{O}_n$. The algebra $\mathcal{O}_n$ is a (basic) example of a C^*-algebra which is *infinite* in the sense that there exists an element v with v^*v equal to a projection, and $vv^* \leq v^*v$ and $vv^* \neq v^*v$.

The following construction is a visualization of this algebra; we call it *the interval picture* for $\mathcal{O}_n$.

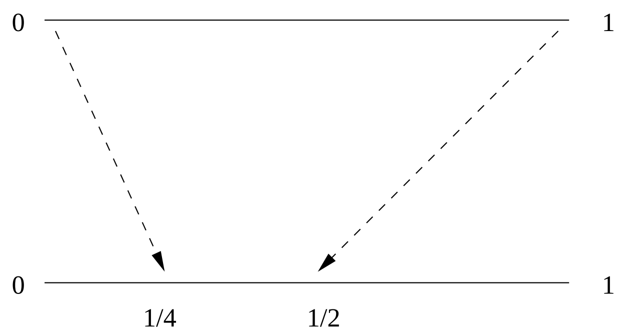


FIGURE 1. Interval picture for the operator S_2S_1 in $\mathcal{O}_2$.

Consider the Hilbert space $H = L^2[0,1]$ and isometries S_i, $i = 1,\ldots,n$ whose ranges are the subspaces of H consisting of functions vanishing a.e. off the interval $[\frac{i-1}{n}, \frac{i}{n}]$. For definiteness we can take

$$S_i f(x) := \begin{cases} \sqrt{n} f(1-i+nx) & \text{if } x \in [\frac{i-1}{n}, \frac{i}{n}] \\ 0 & \text{otherwise.} \end{cases}$$

In other words, $S_i f = \sqrt{n} f \circ h_i^{-1}$ where h_i is the "compression" of $[0,1]$ onto $[\frac{i-1}{n}, \frac{i}{n}]$ given by

$$h_i : [0,1] \to \left[\frac{i-1}{n}, \frac{i}{n}\right] : \ t \mapsto h_i(t) := \frac{i-1+t}{n}.$$

It is clear that $S_i \in \mathcal{B}(L^2[0,1])$ and that $S_i^* f := \frac{1}{\sqrt{n}} f \circ h_i$, for every $f \in L^2[0,1]$. We define $\mathcal{O}_n$ to be the C^*-algebra generated by the isometries S_i, i.e. $C^*(S_1,\ldots,S_n)$. We can easily conclude that

1. $S_iS_i^* = M_{\chi_i}$, the operator of multiplication by the characteristic function χ_i of $[\frac{i-1}{n}, \frac{i}{n}]$,

2. $\sum_{i=1}^n S_iS_i^* = I$.

The subalgebra $\mathcal{F}^n$

An example of a C^*-algebra is the algebra $\mathcal{M}_n(\mathbb{C})$, which has a non-selfadjoint subalgebra consisting of the upper triangular matrices. Another class of examples is given by the UHF C^*-algebras $\mathcal{B}$ with the property

$$\mathcal{M}_{n^k} \subseteq \mathcal{M}_{n^{k+1}} \subseteq \dots \text{ and } \mathcal{B} = \overline{\cup_k \mathcal{M}_{n^k}} \ (= \lim_{\rightarrow} \mathcal{M}_{n^k}).$$

Consider a "word" $\mu = i_1 \dots i_k$ where $i_j \in \{1, 2, \dots, n\}$ and define $S_\mu = S_{i_1} \dots S_{i_k}$, where $|\mu| = k$ is the length of the word. Then the operator S_μ has range a subspace of the form $L^2(E_\mu)$, where E_μ is the range of a succession of compressions of $[0, 1]$.

Proposition 1. *If $|\mu| = |\lambda|$ for two given words μ and λ, then $S_\mu^* S_\lambda$ is I if $\mu = \lambda$ and 0 otherwise.*

Proof. If $\mu = \lambda$ then $S_\mu^* S_\lambda = S_{i_k}^* \dots (S_{i_1}^* S_{i_1}) \dots S_{i_k} = I$. If $\lambda \neq \mu$, and are of equal length, then $E_\mu \cap E_\lambda = \emptyset$ and so $S_\mu^*(S_\lambda f) = 0$, since $S_\lambda f \in L^2(E_\lambda)$. □

When μ and λ have different lengths then, if $\lambda = \mu\lambda'$, $S_\mu^* S_\lambda$ is $S_{\lambda'}$ and if $\mu = \lambda\mu'$, $S_\mu^* S_\lambda$ is $S_{\mu'}$. We can also check that $S_\mu^* S_\lambda$ is 0 in any other case.

Proposition 2. *If $|\mu| = |\lambda|$ for two given words μ and λ, then the operator $S_\mu S_\lambda^*$ has initial space $L^2(E_\lambda)$ and final space $L^2(E_\mu)$.*

We define $\mathcal{F}_k = span\{S_\mu S_\lambda^* : |\mu| = |\lambda| = k\}$ and $\mathcal{F}^n = \overline{\cup_k \mathcal{F}_k}$. Note that $\mathcal{F}_k \simeq \mathcal{M}_{n^k}(\mathbb{C})$, so $\mathcal{F}_k \subseteq \mathcal{F}_{k+1}$ and $\mathcal{F}^n$ is a UHF C^*-subalgebra of $\mathcal{O}_n$.

Let $\mathcal{A}_{n,k}$ be the norm closed unital subalgebra of $\mathcal{O}_n$ generated by the set $S_1, \dots, S_n, S_1^*, \dots, S_k^*$, and let $\mathcal{A}_{n,\emptyset}$ be the norm closed unital algebra generated by $S_1, \dots, S_n$. Then $\mathcal{A}_{1,\emptyset}$, which is generated by a single isometry, is isometrically isomorphic to the disc algebra. For $n \geq 2$, $\mathcal{A}_n \equiv \mathcal{A}_{n,\emptyset}$ is called the non-commutative disc algebra, while $\mathcal{A}_{n,n}$ is, by definition, the C^*-algebra $\mathcal{O}_n$.

Problems:

(a) Show that $\mathcal{O}_n \not\cong \mathcal{O}_m$, iff $n \neq m$,
(b) Show that $\mathcal{A}_n \not\cong \mathcal{A}_m$, iff $n \neq m$,
(c) Show that $\mathcal{A}_{n,k} \not\cong \mathcal{A}_{n,l}$, iff $k \neq l$.

A more general problem is to understand "natural" subalgebras of $\mathcal{O}_n$, for example, those containing the abelian subalgebra

$$\mathcal{C} := \overline{span}\{S_\mu S_\mu^* : \text{ for all } \mu\}. \tag{1}$$

After all, the subalgebras of the complex matrix algebra M_n containing the diagonal matrix units are readily understood.

Fourier series

We define $\mathcal{A} = span\{S_\mu S_\lambda^* :$ for all $\mu, \lambda\}$. This is a $*$-subalgebra of $\mathcal{O}_n$ which is uniformly dense. To see this consider for example, when $|\lambda| \leq |\mu'|$, the product

$$S_\mu S_\lambda^* S_{\mu'} S_{\lambda'}^* = S_\mu [S_\lambda^* S_{i'_1} \ldots S_{i'_k}] S_{i'_{k+1}} \ldots S_{i'_\lambda} S_{\lambda'}^* = S_r S_{\lambda'}^* \text{ or } 0.$$

Moreover, if μ is a word of length k, we can write $S_\mu = S_\mu (S_1^*)^k S_1^k = a S_1^k$, with $a \in \mathcal{F}_k$. In fact, we can see that every word in the operators $S_1, \ldots, S_n, S_1^*, \ldots, S_1^*$ can be rewritten in the form aS_1^k or $(aS_1^k)^* = (S_1^*)^k b$, with $a, b \in \mathcal{F}^n$. In this way we can obtain Formal Fourier series expansions:

Proposition 3. *(i) Each operator a in the $*$-algebra generated by $S_1, ..., S_n$ has a representation*

$$a = \sum_{i=1}^{N} (S_1^*)^i a_{-i} + a_0 + \sum_{i=1}^{N} a_i S_1^i$$

where $a_i \in \mathcal{F}^n$ for each i. This representation is unique if for each $i \geq 1$ we require $a_i = a_i P_i$ and $a_{-i} = P_i a_{-i}$ where P_i is the final projection of S_1^i.
(ii) The linear maps E_i defined by $E_i(a) = a_i$, extend to continuous, contractive, linear maps from $\mathcal{O}_n$ to $\mathcal{F}^n$.
(iii) The generalized Cesaro sums

$$\sigma_k(a) = \sum_{k=1}^{N} (1 - \frac{|k|}{N}) (S_1^*)^k E_{-k}(a) + \sum_{k=0}^{N} (1 - \frac{|k|}{N}) E_k(a) S_1^k$$

converge to a as $N \longrightarrow \infty$.

Proof of (iii). This is the proof of Proposition 9 below. $\square$

PART II

Cuntz algebras, coordinates, subalgebras.

Universal Cuntz Algebras

We now consider a $*$-representation π of the dense subalgebra $\mathcal{A}$ of $\mathcal{O}_n$ given earlier, where $\pi(S_i) = T_i$ and where $T_1, \ldots, T_n$ are isometries, on a separable Hilbert space, such that $T_1T_1^* + ... + T_nT_n^* = I$. We may consider the set of all such representations on separable spaces. We define $\mathcal{O}_n^{univ}$ to be the completion of $\mathcal{A}$ under the universal norm

$$\|a\|_{univ} := \sup\{\|\pi(a)\| : \pi \text{ separably acting } *\text{-representation } \}.$$

In this way we arrive at the algebra in the next definition.

Definition 4. $\mathcal{O}_n^{univ}$ is the universal C^*-algebra generated by n isometries $T_1, ..., T_n$, which satisfy

$$T_1T_1^* + ... + T_nT_n^* = I \tag{2}$$

Take a maximal family $\{H_a\}$ of separable Hilbert spaces and isometries $T_{1,a}, ..., T_{n,a}$ on H_a satisfying (2) and define

$$\widetilde{T_i} := \sum_a \oplus T_{i,a} \quad \text{acting on } \widetilde{H} := \oplus H_a,$$

We may define $\mathcal{O}_n^{univ}$ directly as the generated C^*-algebra $C^*(\widetilde{T_1}, \ldots, \widetilde{T_n})$ acting on $\widetilde{H}$.

We can define *gauge automorphisms* γ_z, $|z| = 1$, of $\mathcal{O}_n^{univ}$ which are given on the generators by $\widetilde{T_i} \to z\widetilde{T_i}$. We claim that each γ_z is isometric. Indeed, for every $*$-representation π of $\mathcal{A}$, one sees that $\pi \circ \gamma_z$ is also a $*$-representation of $\mathcal{A}$; hence, $\|a\|_{univ} \geq \|\pi(\gamma_z(a))\|$ by the definition of $\|\cdot\|_{univ}$. By taking supremum over all π, we get that $\|a\|_{univ} \geq \|\gamma_z(a)\|_{univ}$. Moreover, $\|a\|_{univ} = \|\gamma_{\overline{z}}\gamma_z(a)\|_{univ} \leq \|\gamma_z(a)\|_{univ}$. Thus γ_z is an isometry for every $z \in \mathbb{T}$. Now, since the $z\widetilde{T_i}$ satisfy (2) (it is an easy exercise to show that $\gamma_z(\widetilde{T_i}^*) = \overline{z}\widetilde{T_i}^*$), they generate the universal algebra $\mathcal{O}_n^{univ}$; therefore γ_z maps onto $\mathcal{O}_n^{univ}$.

Note that the universal algebra possesses a UHF subalgebra $\mathcal{F}_{univ}^n$ defined in the same way as before.

Proposition 5. *The map $E_0 : \mathcal{O}_n^{univ} \to \mathcal{F}_{univ}^n$ defined by*

$$E_0(a) := \int_0^1 \gamma_{e^{2\pi it}}(a)dt,$$

where the integral is considered as a Riemann integral of a norm-continuous function, is a contractive, faithful projection. Moreover, if $\mathcal{J}$ is a (closed) ideal of $\mathcal{O}_n^{univ}$, then $E_0(\mathcal{J}) \subseteq \mathcal{J}$.

Remark 6. There is an alternative algebraic definition of E_0 from which it follows that if $\mathcal{J}$ is an ideal of $\mathcal{O}_n^{univ}$, then $E_0(\mathcal{J}) \subseteq \mathcal{J}$.

Theorem 7. *$\mathcal{O}_n^{univ}$ is simple and $\mathcal{O}_n^{univ}$ is isomorphic to $\mathcal{O}_n$. Furthermore, $\mathcal{O}_n$ is isomorphic to every C^*-algebra generated by n isometries $s_1, ...s_n$ satisfying $\sum s_k s_k^* = I$.*

Proof. Take a nonzero ideal $\mathcal{J}$ of $\mathcal{O}_n^{univ}$, then $E_0(\mathcal{J}) \subseteq \mathcal{J}$. But $E_0(\mathcal{J}) \subseteq \mathcal{F}_{univ}^n$. One can show that this implies that $E_0(\mathcal{J})$ contains a "monomial" $X = \widetilde{T_\mu}\widetilde{T_\lambda}^*$ where μ, λ are words with $|\mu| = |\lambda| = k$. Then $T_\mu = XT_\lambda \in \mathcal{J}$, so $I = T_\mu^* T_\mu \in \mathcal{J}$ and so $\mathcal{J} = \mathcal{O}_n^{univ}$.
We define a $*$-representation π of O_n^{univ} by extending the map $\widetilde{T_i} \mapsto S_i$. Then by definition, for $a \in O_n^{univ}$, we have $\|\pi(a)\| \leq \|a\|$. It is clear that π is onto $\mathcal{O}_n$ and because of the simplicity already shown, $\ker \pi = (0)$. So in fact $O_n^{univ} \simeq O_n$. The last conclusion follows by the same arguments. $\square$

Remark 8. For an alternate proof of the simplicity of the universal Cuntz algebra assume the existence of a nonzero $b \in \mathcal{J} \lhd \mathcal{O}_n^{univ}$, and thus the existence of a positive $b^*b = a \in \mathcal{J}$. Recall that E_0 is a contractive, faithful map, and so $E_0(a) \neq 0$. Thus $0 \neq E_0(a) \in \mathcal{J} \cap \mathcal{F}^n$. As in the previous proof $\mathcal{F}^n$ is simple, therefore $J \cap \mathcal{F}^n = \mathcal{F}^n$. Thus $I \in \mathcal{F}_n \subseteq \mathcal{J}$, and $\mathcal{J} = \mathcal{O}_n^{univ}$.

Proposition 9. (1) *Let $a \in \mathcal{O}_n$ and consider the Fourier series*

$$a \sim \sum_{i=0}^{\infty} a_i S_1^i + \sum_{i=1}^{\infty} (S_1^*)^i a_{-i}.$$

Then the coefficients are unique, subject to the property $a_i = a_i P_i$ and $a_{-i} = P_i a_{-i}$, where $P_i = S_1^i (S_1^i)^*$.

(2) *The Cesaro sums $\sigma_N(a)$ of the series converge to a in norm.*

Proof. Fix $a = S_\mu S_\nu^* \in \mathcal{F}_n$ with $|\mu| = |\nu|$ and define $f_a : \mathbb{T} \to \mathcal{O}_n : \ z \mapsto f_a(z) := \gamma_z(a)$. Then, we have $f_a(1) = a$, $\gamma_z(S_1^i) = z^i S_1^i$, hence $\gamma_z(a) = a$. Therefore

$$\gamma_z\big(\sigma_N(a)\big) = \sum_{i=0}^{N} \big(1 - \frac{|i|}{N}\big) a_i z^i S_1^i + \sum_{i=1}^{N} \big(1 - \frac{|i|}{N}\big) (S_1^i)^* a_{-i} z^{-i}.$$

As in the classical case and because of the fact that the $a_i S_1^i$ and $(S_1^i)^* a_{-i}$ are Fourier coefficients of $f_a(z)$ it follows by Cesaro convergence that $\gamma_z\big(\sigma_N(a)\big)$ tends to $f_a(z)$ uniformly in z. By setting $z = 1$ we get the conclusion. $\square$

PART III

Subalgebras of $\mathcal{O}_n$ and graph C^*-algebras.

Three ways to construct subalgebras of $\mathcal{O}_n$

1. Generator constraints.
Let $\mathfrak{S}$ be a semigroup of operators of the form $S_\mu S_\lambda^*$ which includes all the projections $S_\mu S_\mu^*$. We construct a subalgebra by taking the norm-closure of the linear span of $\mathfrak{S}$. This algebra is invariant under the gauge automorphisms of $\mathcal{O}_n$.

2. Fourier series constraints.
Take $A \subseteq \mathcal{F}_n$ triangular, that is, $A \cap A^* = \mathcal{C}$ (see relation (1)). Then

$$\mathcal{A} = \{a \in \mathcal{O}_n : E_0(a) \in A, E_k(a) = 0, k < 0\}$$

is also a triangular subalgebra of $\mathcal{O}_n$.

3. Extrinsic constraints.
Take $\mathcal{N} \subseteq \mathcal{C}$, a maximal totally ordered family of projections. For example, assume $\mathcal{N}$ to consist of all the projections in $[0, k/2^n]$ in the interval picture. Then we get the nest algebra

$$\mathcal{A} = \mathcal{O}_n \cap \operatorname{alg} \mathcal{N} = \{a \in \mathcal{O}_n : (1-p)ap = 0, \forall p \in \mathcal{N}\}.$$

The "Cantorised interval picture" for $\mathcal{O}_n$

The set of infinite paths X is the Cantor set $\prod_{k=1}^{\infty}\{0,1\}$ where every typical path/point is of the form $x = x_1x_2\ldots$ and we consider the space $L^2(X)$ with the natural product measure. Each vertex word $\nu = \lambda_1\lambda_2...\lambda_k$ defines an "interval" E_ν in X of points x which start with ν, that is

$$E_\nu := \{x = \lambda_1\lambda_2...\lambda_k x_1 x_2...\}.$$

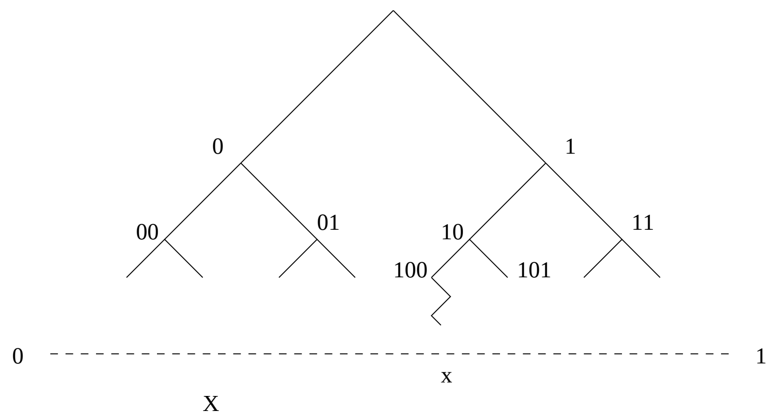


Figure 2. The Cantorised interval picture.

We define

$$\alpha_{\mu,\nu} : E_\nu \to E_\mu : \nu x_1 x_2 ... \mapsto \mu x_1 x_2 ...$$

For example

$$\alpha_{0,\emptyset} : x \mapsto 0x \text{ and } \alpha_{1,\emptyset} : x \mapsto 1x.$$

Then $S_i f\big(a_{0,\emptyset}(x)\big) = \sqrt{2} f(x)$, $f \in L^2(X)$. Thus $\mathcal{O}_n$ acts on $L^2(X)$. We have

$$\mathcal{C} := C^*\{S_\mu S_\mu^* : \forall \mu\} \simeq C(X),$$

for example $S_1 S_1^* = M_{x_E}$, where $E = \{x : x_1 = 1\}$. We define the support of an operator a in $\mathcal{F}^n$ or $\mathcal{O}_n$ as a subset of $X \times X$. We can view this as a cantorized picture for the support.
We define the topological binary relation (or groupoid) for $\mathcal{F}_n$ to be the subset of $X \times X$,

$$R(\mathcal{F}^n) := \cup\{graph\big(\alpha_{\mu,\nu}\big) : |\mu| = |\nu|\},$$

where $graph\big(\alpha_{\mu,\nu}\big) := \{\big(\alpha_{\mu,\nu}(x), x\big) : x \in E_\nu\} \equiv E_{\mu,\nu}$. This is an equivalence relation, considered with the topology having $\{E_{\mu,\nu}\}$ as a basis of open-closed sets. We also define $R(\mathcal{O}_n) := \{(x,k,y) : \alpha_{\mu,\nu}(y) = x, |\mu| - |\nu| = k\}$. Then, we can easily check that $(x,k,y)^{-1} = (y,-k,x)$, under the (partially defined) multiplication $(x,k,y)(y,k',z) = (x,k+k',z)$, which turns $R(\mathcal{O}_n)$ into a groupoid.

In general, if $A \subseteq \mathcal{F}^n$, we can define $R(A) = \cup\{E_{\mu,\nu} : S_\mu S_\nu^* \in A, |\mu| = |\nu|\}$. If, in particular A is a (closed) subalgebra, then $R(A)$ is a transitive binary relation.

Theorem 10. *If $\mathcal{C} \subseteq A \subseteq \mathcal{F}^n$ and A is a closed subalgebra, then the topological binary relation $R(A)$ is a complete invariant for isometric isomorphisms. In other words, $A \simeq A'$ isometrically, if and only if, $R(A)$ is topologically isomorphic to $R(A')$ by a binary relation isomorphism.*

Theorem 11. *If $A, A' \subseteq \mathcal{O}_n$ are triangular norm-closed subalgebras and are gauge invariant (i.e. $\gamma_z(A) \subseteq A$ for all $|z| = 1$), then*

(1) *A (and A') is generated by operators of the form $S_\mu S_\nu^* \in A$,*
(2) *A and A' are isometrically isomorphic if and only if $R(A) \simeq R(A')$ in the sense of topological semigroupoids.*

Normalizing partial isometries

Definition 12. A partial isometry U in $\mathcal{F}^n$ (or, in general, in $\mathcal{O}_n$) is called $\mathcal{C} - normalising$ if $U\mathcal{C}U^* \subseteq \mathcal{C}$ and $U^*\mathcal{C}U \subseteq \mathcal{C}$.

Examples are: any matrix unit $S_\mu S_\nu^*$, $|\mu| = |\nu| = k$, certain sums of matrix units, and elements of the form UD (or DU), where U is $\mathcal{C}$-normalising and $D \in \mathcal{C}$.

Theorem 13. *Let $U \in \mathcal{F}^n$. The following are equivalent*

(1) *U is a $\mathcal{C}$-normalising partial isometry.*
(2) *U is the orthogonal finite sum of partial isometries of the form $DS_\mu S_\nu^*$, $|\mu| = |\nu|$ and $D \in \mathcal{C}$.*
(3) *$\|QUP\| = 0$ or 1, for every projection $P, Q \in \mathcal{C}$.*

The key fact in this proof is that an element $b \in F^n$ can be approximated by elements $\Delta_m(b)$ in small explicit subalgebras $\widetilde{\mathcal{F}_m^n}$. Indeed, for an element $b \in \mathcal{F}^n$ the diagonal part $\Delta(b) \in \mathcal{C}$ can be defined as the limit of block diagonal matrices $b_k := \sum_i e_{ii}^k b e_{ii}^k$, as k tends to infinity, where the e_{ii}^k are the diagonal matrix units in $\mathcal{F}_k^n$. Then the map $\Delta : \mathcal{F}^n \to \mathcal{C}$ is a faithful projection. For a fixed m we can use block maps (through matrix unit projections of the commutants of $\mathcal{F}_m^n$ and $\mathcal{F}_k^n$ for $k = m, m+1...$) to define explicit maps $\Delta_m : \mathcal{F}^n \to \widetilde{\mathcal{F}_m^n}$, where $\widetilde{\mathcal{F}_m^n} = C^*(\mathcal{C}, \mathcal{F}_m^n)$ (this is the algebra $\mathcal{F}_m^n \otimes (e_{ii}^m \mathcal{C} e_{ii}^m)$). Thus $\Delta_m(b) \longrightarrow b$, $\forall\, b \in \mathcal{F}^n$.

Proof of the theorem. It is easy to check that (2) $\Rightarrow$ (1) $\Rightarrow$ (3). So it suffices to show that (3) $\Rightarrow$ (2). Take U satisfying the 0-1 property and choose m big enough so that $U = \Delta_m(U) + U'$, $\|U'\| < 1$. Observe that U' and $\Delta_m(U)$ satisfy the 0-1 property, too. The implication (3) $\Rightarrow$ (2) is straightforward for the elements of $\widetilde{\mathcal{F}_k^n}$. Thus it remains to show that any U' with the 0-1 property and $\|U'\| < 1$ is necessarily 0. But this can be proved by taking $P = Q = I$. $\square$

PART IV

The C^*-envelope of an operator algebra

Definition 14. Let $\mathcal{A}$ and $\mathcal{B}$ be operator algebras. A map

$$\phi : \mathcal{A} \to \mathcal{B}$$

is called a complete isometry if the maps

$$\phi^{(n)} : M_n(\mathcal{A}) \to M_n(\mathcal{B}) : (a_{ij}) \mapsto \big(\phi(a_{ij})\big)$$

are isometries for every $n \in \mathbb{N}$.

A way to define the C^*-envelope of an algebra is through Hamana's theory or by using boundary ideals (or Shilov ideals) defined in the following result.

Proposition 15. *Let $\mathcal{A}$ be an operator algebra and $\mathcal{C} := C^*(\mathcal{A})$. There is a closed ideal J of $\mathcal{C}$ so that*

(1) *The map $\mathcal{A} \to \mathcal{C}/J$ is a complete isometry (J is called a boundary ideal),*
(2) *The ideal J contains all boundary ideals (J is called a Shilov ideal).*

We call $\mathcal{C}/J$ the *C^*-envelope of $\mathcal{A}$* and denote it by $C_{env}(\mathcal{A})$. An alternative definition for the C^*-envelope of an algebra $\mathcal{A}$ is given through the following universal property:

For every complete isometry π of $\mathcal{A}$ into a C^*-algebra $\mathcal{B}$ such that $C^*(\pi(\mathcal{A})) = \mathcal{B}$, there exists a $*$-morphism ϕ of $\mathcal{B}$ onto $C_{env}(\mathcal{A})$, so that $\phi(\pi(a)) = a$, for every $a \in \mathcal{A}$.

Proposition 16. *The two definitions are equivalent.*

For a "modern" proof of this fact see Arveson's notes at his website.

Example 17. If we start with an operator algebra $\mathcal{A}$ and the $*$-algebra $C^*(\mathcal{A})$ is simple, then $C_{env}(\mathcal{A}) = C^*(\mathcal{A})$ since the only (maximal Shilov) ideal is (0).

Example 18. If an operator algebra $\mathcal{A}$ is generated by unitaries, then $C_{env}(\mathcal{A}) = C^*(\mathcal{A})$.

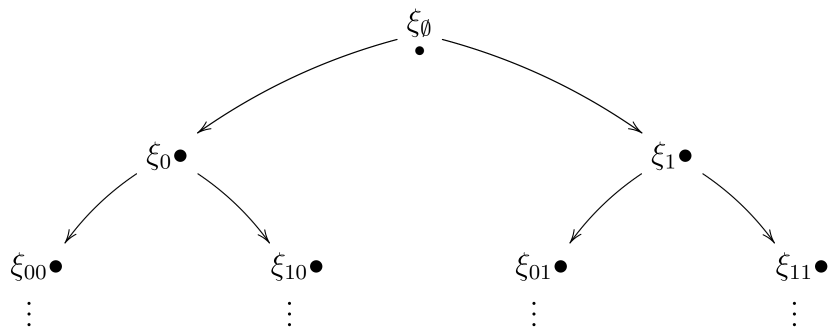
Analytic Toeplitz operators

Let L be the unilateral shift defined on the usual basis of l^2 by $Le_n := e_{n+1}$. Then, the normed closed algebra generated by L and I, i.e. $Alg(L, I)$, is isomorphic to the disc algebra $A(\mathbb{D})$ and $\overline{Alg(L, I)}^{w^*}$ are the analytic Toeplitz operators. Moreover $C^*(L)$ contains the compact operators and is called the Toeplitz C^*-algebra and $C^*(L)/\mathcal{K}(\mathcal{H}) \simeq C(\mathbb{T})$ (the quotient map is faithful on $Alg(L, I)$). For the unilateral shift we can have a graph picture:

$$e_0 \to e_1 \to e_2 \cdots$$

Cuntz-algebras and Cuntz-Toeplitz algebras

We consider, now, the case of a system of two unilateral shifts and the graph picture:



Then $L_0L_0^* + L_1L_1^* = I - \xi_\emptyset \otimes \xi_\emptyset$, which resembles the defining relation of the Cuntz algebra $\mathcal{O}_2$. In general, if we consider n isometries $L_0, ..., L_{n-1}$, given by $L_i\xi_w = \xi_{iw}$, then $\sum L_iL_i^* = I - \xi_\emptyset \otimes \xi_\emptyset$ (with orthogonal ranges). The norm-closed unital algebra generated by $L_0, ..., L_{n-1}$, i.e. $Alg(I, L_0, ..., L_{n-1})$, is the non-commutative disc algebra $\mathbb{A}_n$. The generated C^*-algebra is called the Cuntz-Toeplitz algebra $\mathcal{TO}_n$. We end up with the corresponding Cuntz algebra $\mathcal{O}_n$ by taking the quotient of $C^*(I, L_0, ...L_{n-1})$ by $\mathcal{K}(\mathcal{H})$, for in that case $\sum \widetilde{L}_i\widetilde{L}_i^* = I$.

Consider a finite word w consisting of letters in $\{0, 1..., n-1\}$ and take the corresponding vector ξ_w. For any word i, define the operator

$$R_i : \mathcal{H} \to \mathcal{H} : \xi_w \mapsto R_i\xi_w := \xi_{wi}.$$

Then $R_iL_j\xi_w = R_i\xi_{jw} = \xi_{jwi} = L_j\xi_{wi} = L_jR_i\xi_w$. Thus, $R_iL_j = L_jR_i$ (note also that $R_0R_0^* + ... + R_{n-1}R_{n-1}^* = I - \xi_\emptyset \otimes \xi_\emptyset$). Hence we have $\overline{Alg(I, R_0, ...R_{n-1})}^{w^*} \subseteq \mathbb{A}_n'$. In fact, $\overline{Alg(I, R_0, ...R_{n-1})}^{w^*} = \mathbb{A}_n'$. Moreover, $\mathbb{A}_n'' = \overline{Alg(I, L_0, ..., L_{n-1})}^{w^*} \equiv \mathcal{L}_n$.

The C^*-envelope of $\mathbb{A}_n$

Proposition 19. *Let A be an operator algebra and suppose that A' contains a sequence of isometries R_n, such that $R_n \overset{WOT}{\longrightarrow} 0$. Then, the restriction of the Calkin map $\pi : A \to A/\mathcal{K}(\mathcal{H})$ to A is an isometry.*

Proof. The key fact in the proof is that, if $R_n \overset{WOT}{\longrightarrow} 0$, then $KR_n \overset{SOT}{\longrightarrow} 0$, for every $K \in \mathcal{K}(\mathcal{H})$. Indeed, if K is a rank one operator, so that $K = e \otimes f$, for some vectors e, f, then $\|KR_n\xi\| = |\langle R_n\xi, f\rangle| \|e\| \longrightarrow 0$. By

linearity the same holds for finite rank operators. Let $K \in \mathcal{K}(\mathcal{H})$, then for every $\epsilon > 0$ there exists a finite rank operator F such that $\|K - F\| < \epsilon$. Then $\|KR_n\xi\| \leq \|(K-F)R_n\xi\| + \|FR_n\xi\| < \epsilon\|R_n\xi\| + \|FR_n\xi\| = \epsilon\|\xi\| + \|FR_n\xi\| \longrightarrow \epsilon\|\xi\|$. Thus, $\|KR_n\xi\| \longrightarrow 0$.
We want to prove that the Calkin map is an isometry. For every unit vector $\xi \in \mathcal{H}$, we have

$$\begin{aligned}\|A\xi\| = \|R_nA\xi\| = \|AR_n\xi\| &\leq \|(A+K)R_n\xi\| + \|KR_n\xi\| \\ &\leq \|A+K\| + \|KR_n\xi\|.\end{aligned}$$

So, by using the previous assertion, we get that $\|A\| \leq \|A+K\|$, and thus $\|A\| \leq \|A + \mathcal{K}(\mathcal{H})\|$. On the other hand, by definition $\|A + \mathcal{K}(\mathcal{H})\| = \inf\{\|A+K\| : K \in \mathcal{K}(\mathcal{H})\|\} \leq \|A\|$. $\square$

Corollary 20. *The Calkin map on $C^*(L_0, ...L_{n-1})$ is a complete isometry on $Alg(I, L_0, ...L_{n-1})$.*

Proof. It suffices to show that $Alg(I, L_o, ...L_{n-1})' = \mathbb{A}_n'$ contains a sequence $\{R_n\}_n$ that converges in the weak operator topology to 0. For this reason we set $R_n = R_1^n$ and note that for every w', there is n_o so that $\langle R_1^n \xi_w, \xi_{w'}\rangle = 0$, for every $n \geq n_0$. Since $\{R_1^n\}$ is uniformly bounded, we get that R_n converges to 0 in the weak operator topology. $\square$

To sum up, we began with a Cuntz-Toeplitz algebra $\mathcal{TO}_n$ which contains all the rank one operators (since it contains $\xi_\emptyset \otimes \xi_\emptyset$), hence the whole ideal $\mathcal{K}(\mathcal{H})$. Then $\mathcal{TO}_n/\mathcal{K}(\mathcal{H})$ is a Cuntz algebra, by what we have shown about the universality of $\mathcal{O}_n^{univ}$. The restriction of the Calkin map on $\mathbb{A}_n$ is a complete isometry (corollary 20), and thus $\mathbb{A}_n$ embeds isometrically in a simple C^*-algebra, that is generated by $\mathbb{A}_n$. Hence, $C_{env}(\mathbb{A}_n) = \mathcal{TO}_n/\mathcal{K}(\mathcal{H}) = \mathcal{O}_n^{univ}$. Note also, that in this case $\pi(\mathbb{A}_n)$ is isomorphic to $\mathcal{A}_{n,\emptyset}$, so the non-commutative disc algebras of each case are identified (modulo $\mathcal{K}(\mathcal{H})$).

Part V

Graph algebras

Let G be a directed graph with G_0 the set of vertices (denoted by letters $x, y, \ldots$) and G_1 the set of directed edges (denoted by letters $e, f, \ldots$). In order to make matters simple, we assume that our graphs are finite, i.e., both G_0 and G_1 are finite, and that there are no sources. Certainly the main results about C^*-envelopes are valid for arbitrary graphs (actually in much greater generality by a recent result of Katsoulis and Kribs).

The family of all finite directed paths (even the trivial one of length 0) in G is denoted by $\mathbb{F}^+(G)$. The path

$$x \xrightarrow{e_1} y \xrightarrow{e_2} \cdots z$$

is denoted by the sequence $z \cdots e_2 e_1 x$. We consider the space with basis $\mathbb{B} = \{\xi_w : w \in \mathbb{F}^+(G)\}$ with the obvious inner product; we get the Hilbert space $\mathcal{H}_G$, which is separable in the case of countably many vertices. Define the operators

$$L_u : \mathcal{H}_G \to \mathcal{H}_G : \ \xi_w \mapsto L_u \xi_w := \begin{cases} \xi_{uw} \text{ , if } r(w) = s(u) \\ 0 \text{ , otherwise} \end{cases}$$

where $s(u)$ denotes the beginning of the path u and $r(w)$ denotes the end of the path w. Moreover, for a vertex $x \in G_0$, we define P_x to be the projection onto $\{\xi_w : r(u) = x\}$, namely L_x. We call the space $P_{s(u)}$ *the initial projection of* L_u. We define

$$\mathcal{A}_G := Alg\{I, L_u, u \in \mathbb{F}^+(G)\} = Alg\{I, L_e, P_x : e \in G_1, x \in G_0\}.$$

The objective of this section is to calculate the C*-envelope of $\mathcal{A}_G$.

Let's start with a few examples for $\mathcal{A}_G$.

Example 21. Let $C_n = \{x, e_0, \ldots e_{n-1}\}$. Then $\mathcal{A}_{C_n} = \mathcal{A}_n$.

Example 22. Let G be the graph

$$x \xrightarrow{e} y$$

Then $\mathcal{A}_G = T_2$. Indeed, since $\mathbb{B} = \{\xi_x, \xi_e, \xi_y\}$, we have

$$\xi_x \xrightarrow{L_e} \xi_e$$

Moreover, relative to the basis $\{\xi_x, \xi_e, \xi_y\}$ of $\mathbb{B}$, we obtain

$$L_e = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, P_x = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, P_y = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

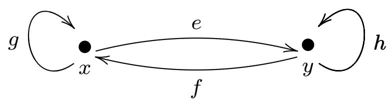
By compressing on the subspace generated by ξ_x, ξ_y, we obtain an isomorphism of $\mathcal{A}_G$ onto T_2.

Similarly, if one considers a graph G of the form

$$x_1 \to x_2 \to \cdots \to x_n,$$

then the corresponding algebra $\mathcal{A}_G$ is isomorphic to T_n.

Example 23. What is the representation in more complex cases? For example, let G be the graph



Each path corresponds to a sequence of x's and y's with

$$\begin{gathered} x \to x \text{ corresponding to } L_g \text{ , i.e. } P_x \\ x \to y \text{ corresponding to } L_e \\ y \to x \text{ corresponding to } L_f \\ y \to y \text{ corresponding to } L_h \text{ , i.e. } P_y \end{gathered}$$

Thus, $\mathcal{A}_G = Alg\{I, L_e, L_f, L_g, L_h, P_x, P_y\}$. By the time we prove that $C_{env}(\mathcal{A}_G)$ is $C^*(G)$ (the Cuntz-Krieger algebra), we will see how the theory of Cuntz-Toeplitz algebras can be helpful, in these complex cases. In particular, one can verify that $\mathcal{A}_G$ is the non-selfadjoint operator algebra generated by two Cuntz isometries and their range projections.

By using the very definitions we can prove the following properties for L_e, $e \in G_1$ and P_x, $x \in G_0$.
1. $L_e^* L_f = 0$, *for* $e \neq f$: Observe that

$$L_e^* : \mathcal{H}_G \to \mathcal{H}_G : \ \xi_w \mapsto L_e^* \xi_w := \begin{cases} \xi_{w'} \text{ , if } w = ew' \\ 0 \text{ , otherwise} \end{cases} .$$

So, if we take a path w with $r(w) = s(f)$, we get $L_e^* L_f(\xi_w) = L_e^*(\xi_{fw}) = 0$.
2. $P_x P_y = 0$, *for* $x \neq y$.
3. $L_e^* L_e = P_{s(e)}$. (Just note that $L_e^* L_e$ is the initial projection of L_e.)
4. $\sum_{r(e)=x} L_e L_e^* \leq P_x$, *for* $x \in G_0$. (As a matter of fact $\sum L_e L_e^* < P_x$, because $L_e L_e^* \xi_e = L_e \xi_\emptyset = \xi_\emptyset \neq \xi_e$.)

These properties are called the Cuntz-Krieger-Toeplitz relations. A more restrictive set of relations becomes the Cuntz-Krieger relations, namely

(1) $s_e^* s_f = 0$, for $e \neq f$,
(2) $q_x q_y = 0$, for $x \neq y$,
(3) $s_e^* s_e = q_{s(e)}$, $e \in G$,
(4) $\sum_{r(e)=x} s_e s_e^* = q_x$, for $x \in G_0$,
(5) $s_e s_e^* \leq q_x$, for $r(e) = x$ (and, therefore, the isometries s_e have mutual orthogonal ranges).

The C^*-algebra of a graph G (denoted $C^*(G)$) generated by a universal family $\{Q_x, S_e\}$ that satisfy the Cuntz-Krieger relations is called *the Cuntz-Krieger C^*-algebra of the graph G* and is denoted by $C^*(G)$.

The main result is the following.

Theorem 24. *$C_{env}(\mathcal{A}_G) \simeq C^*(\mathcal{A}_G)/\mathcal{K}(\mathcal{H}) \simeq C^*(G)$.*

Corollary 25. *Let $G = \{x, e_0, ...e_{n-1}\}$ then $C_{env}(\mathcal{A}_G) \simeq C^*(G)$, thus $C_{env}(\mathcal{A}_n) \simeq \mathcal{O}_n$.*

The proof of Theorem 24 will be similar to the one in the previous section but in our case the resulting envelopes are not simple. We therefore need to gain an understanding about their ideals, which requires the use of selfadjoint theory.

The gauge invariance uniqueness theorem

Suppose the families $\{\widehat{Q_x}\}_{x\in G_0}$, $\{\widehat{S}_e\}_{e\in G_1}$ satisfy the Cuntz-Krieger relations. We say that the Cuntz-Krieger algebra $C^*(\widehat{Q_x}, \widehat{S}_e)$ admits a *gauge action* if the maps $\hat{\beta}_z$, $z \in \mathbb{T}$ given by $\hat{\beta}_z(\widehat{Q_x}) = \widehat{Q_x}$ and $\hat{\beta}_z(\widehat{S}_e) = z\widehat{S}_e$ extend to $*$-automorphisms of $C^*(\widehat{Q_x}, \widehat{S}_e)$.

Example 26. The universal algebra $C^*(G)$ admits a gauge action, which we denote as β_z, $z \in \mathbb{T}$.

Example 27. The algebra $C^*(\mathcal{A}_G)/\mathcal{K}(\mathcal{H})$ admits a gauge action α_z. We begin by defining

$$u_z : \mathcal{H}_G \to \mathcal{H}_G : \xi_w \mapsto u_z(\xi_w) := \overline{z}^{|w|}\xi_w,$$

where $|w|$ is the length of the path. Then $u_z^* L_e u_z = zL_e$ and $u_z^* P_x u_z = P_x$. Indeed, $u_z^*(\xi_w) = z^{|w|}\xi_w$, therefore

$$u_z^* L_e u_z(\xi_w) = \overline{z}^{|w|} u_z^* L_e \xi_w = \overline{z}^{|w|} u_z^* \xi_{ew} = \overline{z}^{|w|} z^{|we|} \xi_{ew} = z\xi_{ew} = zL_e\xi_w$$

if $r(w) = s(e)$ and $u_z^* L_e u_z(\xi_w) = 0$ otherwise. Thus, defining $\alpha_z(T) = u_z^* T u_z$ we get $\alpha_z(L_e) = zL_e$. Moreover, $u_z^* P_x u_z(\xi_w) = \overline{z}^{|w|} u_z^* P_x(\xi_w) = \overline{z}^{|w|}\overline{z}^{|xw|}\xi_{xw} = \xi_{xw} = P_x(\xi_w)$ and 0 otherwise. Thus, $\alpha_z(P_x) = u_z^* P_x u_z = P_x$. Besides, since $|z| = 1$, the α_z are isometric automorphisms of $\mathcal{B}(\mathcal{H}_G)$ and map the linear span of the generators onto itself; hence they induce automorphisms of $C^*(\mathcal{A}_G)/\mathcal{K}(\mathcal{H})$.

Note that this example is not valid if G has sources, which explains some of the difficulties arising in the general case.

In order to prove the gauge invariance uniqueness theorem, we study an expectation associated with the gauge action.

Suppose that the Cuntz-Krieger algebra $C^*(\widehat{Q_x}, \widehat{S}_e)$ admits a gauge action $\hat{\beta}_z$, $z \in \mathbb{T}$. We define the map

$$\widehat{\Phi}(A) = \int_0^{2\pi} \hat{\beta}_{e^{it}}(A) \frac{dt}{2\pi}, \quad A \in C^*(\widehat{Q_x}, \widehat{S}_e),$$

as a Riemann integral. Notice that $\widehat{\Phi}$ is contractive, positive and faithful, i.e., there are no non-zero positive elements in its kernel. With the aid of this map $\widehat{\Phi}$, we identify the fixed point algebra $\mathcal{F}_G$ of $\{\hat{\beta}_z, z \in \mathbb{T}\}$, i.e., the subalgebra of $C^*(\widehat{Q_x}, \widehat{S}_e)$ generated by the common fixed points of $\{\hat{\beta}_z, z \in \mathbb{T}\}$.

Definition 28. If G is a directed graph and $C^*(\widehat{Q_x}, \widehat{S}_e)$ a Cuntz-Krieger algebra for that graph, then $\widehat{\mathcal{F}}_{G,n}$, $n \in \mathbb{N}$, will denote the subalgebra of $C^*(\widehat{Q_x}, \widehat{S}_e)$ generated by all elements of the form $\widehat{S}_w \widehat{S}_u^*$, $|w| = |u| = n$.

We now have

Lemma 29. *Suppose that the Cuntz-Krieger algebra $C^*(\widehat{Q_x}, \widehat{S}_e)$ associated with a directed graph G admits a gauge action $\hat{\beta}_z$, $z \in \mathbb{T}$. Then,*

$$\widehat{\mathcal{F}}_G = \overline{\bigcup_n \widehat{\mathcal{F}}_{G,n}}.$$

Proof. Clearly,

$$\widehat{\mathcal{F}}_G \supseteq \overline{\bigcup_n \widehat{\mathcal{F}}_{G,n}}.$$

For the reverse inclusion, notice that the range of $\widehat{\Phi}$ contains (actually coincides with) $\widehat{\mathcal{F}}_G$. So it suffices to prove the result for elements of the form $\widehat{\Phi}(A)$. However, if $\widehat{\Phi}$ is applied to any polynomial on the generators $\widehat{Q_x}, \widehat{S}_e$, then it maps it to $\bigcup_n \widehat{\mathcal{F}}_{G,n}$. Hence the conclusion follows from an easy approximation argument. □

As a corollary, notice that the algebras $\widehat{\mathcal{F}}_{G,n}$, $n \in \mathbb{N}$, form an increasing chain of finite dimensional C^*-algebras, with central projections $\widehat{P}_x$, and so $\widehat{\mathcal{F}}_G$ is an AF C^*-algebra.

Theorem 30. *(gauge invariance uniqueness) If the algebra $C^*(\widehat{Q_x}, \widehat{S}_e)$ admits a gauge action and $\widehat{Q}_x \neq 0$, for every $x \in G_0$, then the map*

$$Q_x \mapsto \widehat{Q_x}, \quad S_e \mapsto \widehat{S}_e$$

extends to a $*$-isomorphism $\psi : C^*(G) \longrightarrow C^*(\widehat{Q_x}, \widehat{S}_e)$.

Proof. Clearly such a $*$-homomorphism ψ exists; the issue is to show that its kernel is trivial. Since $\widehat{Q}_x \neq 0$, for every $x \in G_0$, ψ does not annihilate the central projections of all $\widehat{\mathcal{F}}_{G,n}$, $n \in \mathbb{N}$. Therefore by inductivity of ideals in AF algebras, ψ is injective on $\mathcal{F}_G \subseteq C^*(G)$. Now the fact that $C^*(\widehat{Q_x}, \widehat{S_e})$ admits a gauge action implies that $\hat{\beta}_z \circ \psi = \psi \circ \beta_z$, $z \in \mathbb{T}$. Hence,

$$\widehat{\Phi} \circ \psi = \psi \circ \Phi.$$

Assume by way of contradiction that there exists a positive element $A \in C^*(G)$ so that $\psi(A) = 0$. By the above identity, $\psi(\Phi(A)) = 0$. But this contradicts the fact that ψ is injective on $\mathcal{F}_G$. $\square$

Corollary 31. *If $\mathcal{J} \subseteq \mathcal{C}^*(G)$ is a non-zero gauge invariant ideal then it contains one of the projections Q_x.*

Proof. If not then by the above theorem, the natural quotient map would be an isomorphism. $\square$

Now here is the proof of Theorem 24: The Calkin map

$$\mathcal{A}_G \subseteq C^*(\mathcal{A}_G) \to C^*(\mathcal{A}_G)/\mathcal{K}(\mathcal{H}) \simeq C^*(G)$$

is completely isometric on $\mathcal{A}_G$ (by the same proof as in the case of $\mathcal{A}_n$). If there exists a Shilov ideal, say J, then it will be invariant under the gauge action, because the Shilov ideal contains all boundary ideals and $\mathcal{A}_G$ is invariant under the gauge action. If $J \neq 0$, then $\beta_z(J) \lhd J$, therefore by Corollary 31 J contains projections Q_x (since J is a boundary ideal as well). This leads to a contradiction. $\square$

Part VI

Higher rank algebras

The algebra $\mathbb{A}_n$

Let $\mathbb{F}_n^+$ be the free semigroup generated by $e_1, \ldots, e_n$ and $\mathcal{H}_n = l^2(\mathbb{F}_n^+)$ the Hilbert space with a basis $\{\xi_w : w \in \mathbb{F}_n^+\}$, where w is a word on the letters $e_1, \ldots, e_n$ (we allow the empty word). We set $L_i \equiv L_{e_i}$ the shift operator such that

$$L_i : \mathcal{H}_n \to \mathcal{H}_n : \xi_w \mapsto \xi_{e_i w}.$$

Observe that $L_1 L_1^* + \cdots + L_n L_n^* = I - P_\emptyset$ and define $\mathbb{A}_n = Alg\{I, L_1, \ldots, L_n\}$. We have shown that

$$C^*(\mathbb{A}_n) \supseteq \mathcal{K}(\mathcal{H}) \text{ and } C^*(\mathbb{A}_n)/\mathcal{K}(\mathcal{H}) \simeq \mathcal{O}_n,$$

so $C_{env}(\mathbb{A}_n) \simeq C^*(\mathcal{A}_n) = C_{env}(\mathcal{A}_n)$.

The algebra $\mathcal{A}_\theta$

Fix integers m, n and a permutation θ of the set

$$\{(i,j) : 1 \leq i \leq n, 1 \leq j \leq m\}.$$

We define $\mathbb{F}_\theta^n$ to be the set generated by $\emptyset, e_1, \ldots, e_n, f_1, \ldots, f_m$ subject to $e_i f_j = f_{j'} e_{i'}$, where $\theta(i,j) = (i', j')$. Every element in $\mathbb{F}_\theta^n$ may be written in a reduced form $w = w_e w_f$, where $w_e \in \mathbb{F}_n^+$ (a word in e_i) and $w_f \in \mathbb{F}_m^+$ (a word in f_j). We define the (higher rank) degree of the word w to be $(|w_e|, |w_f|) \in \mathbb{Z}_+^2$. Observe that the uniqueness of the reduced word w follows because θ is a permutation. As in the case of a discrete cancellative semigroup $\mathfrak{S}$, we define the Fock space $l^2(\mathbb{F}_\theta^+)$ and the higher rank non-commutative disc algebra $\mathcal{A}_\theta$ that is generated by the shifts $I, L_{e_1}, \ldots, L_{e_n}, L_{f_1}, \ldots, L_{f_m}$.

Graphs

The semigroup $\mathbb{F}_\theta^+$ can be considered as the set of paths of the two-coloured graph containing one vertex x and a set of blue edges $e_1, ..., e_n$ together with a set of red edges $f_1, ..., f_m$, where some of the blue/red paths are equivalent through the commutation relations given by θ. So, we obtain a generalization of the algebra $\mathcal{A}_n$, using a directed graph to define the generators.

Some of the problems that occur in recent research are the following:

(1) Reflexivity of invariant subspaces for $\mathcal{L}$ (Kribs-Power).
(2) Classification of $\mathcal{A}_\theta$ (Power, Power-Solel).
(3) Representations (Davidson-Power-Yang).
(4) Dilation theory (Davidson-Power-Yang).

(5) Automorphisms (Power-Solel).

Eigenvectors and character space

We define $V_\theta = \{(\underline{z}, \underline{w}) \in \mathbb{C}^n \times \mathbb{C}^m : z_i w_j = w_{j'} z_{i'}, (i', j') = \theta(i,j)\}$, and $\Omega_\theta := V_\theta \cap (\overline{\mathbb{B}_n} \times \overline{\mathbb{B}_m})$. We recall that the disc algebra $\mathcal{A}_1$ (generated by the shift operator and I) has character space $\mathfrak{M}(\mathcal{A}_1) \simeq \overline{\mathbb{B}_1} = \sigma(S)$, with interior $\mathbb{B}_1$ consisting of the eigenvalues of the backward shift $S^*(\sum \overline{a}^n e_n) = \overline{a}(\sum \overline{a}^n e_n)$. One can prove, also, that $\mathfrak{M}(\mathcal{A}_n) \simeq \overline{\mathbb{B}_n}$.

Proposition 32. $\mathfrak{M}(\mathcal{A}_\theta) \simeq \Omega_\theta$.

Remark 33. We will prove the proposition in the case of $\mathcal{A}_n$. Observe that $\begin{bmatrix} L_1 & \cdots & L_n \end{bmatrix} : \mathcal{H} \oplus \cdots \oplus \mathcal{H} \to H$ is a contraction. Then $\begin{bmatrix} L_1 & \cdots & L_n \end{bmatrix} \begin{bmatrix} a_1 I \\ \vdots \\ a_n I \end{bmatrix}$ is also a contraction of $\mathcal{B}(\mathcal{H})$ for all $a = (a_1, \cdots, a_n) \in \mathbb{B}_n$, so $\|a_1 L_1 + \cdots + a_n L_n\| \leq 1$. If $\varphi \in \mathfrak{M}(\mathcal{A}_\theta)$, then $|\varphi(a_1 L_1 + \cdots + a_n L_n)| \leq 1$. But if $|a_1 z_1 + \cdots a_n z_n| \leq 1$, for every $a = (a_1, \cdots, a_n) \in \mathbb{B}_n$ then $(z_1, \cdots, z_n) \in \mathbb{B}_n$. Thus, $\mathfrak{M}(\mathcal{A}_n) \subseteq \overline{\mathbb{B}_n}$ through $\varphi \to (\varphi(L_1), \cdots, \varphi(L_n)) = \underline{z}$. For the opposite direction, one can construct eigenvectors $\lambda_{\underline{z}}$ for $L_1^*, \cdots, L_n^*$, with $L_i^* \lambda_{\underline{z}} = \overline{z_i} \lambda_{\underline{z}}$.

Corollary 34. *$\mathcal{A}_n$ and $\mathcal{A}_m$ are not isometrically isomorphic operator algebras for $n \neq m$. Moreover the quotient algebra $\mathcal{A}_n / com\mathcal{A}_n$ (where $com\mathcal{A}_n$ is the commutator ideal of $\mathcal{A}_n$) is isomorphic to a function algebra $\mathcal{A} \subseteq \mathcal{A}(\mathbb{B}_n)$ (the ball algebra).*

$\mathcal{A}_n$ is the n-shift function algebra of Arveson, so we get connections to complex analysis. We have similar results for $\mathcal{L}_n = \overline{\mathcal{A}_n}^{w^*}$.

Classification

Theorem 35. *For $n = m = 2$ there are 9 non-isomorphic algebras $\mathcal{A}_\theta$, determined by the 24 permutations θ of the set $\{(i,j) : 1 \leq i \leq j \leq 2\}$.*

The 24 permutations give 9 classes of semigroups $\mathcal{F}_\theta^+$, with representative permutations $\theta_1, \ldots, \theta_9$. It is natural to ask how many isomorphism types there are for the corresponding algebras $\mathcal{A}_\theta$. By using the Gelfand space one can separate all of these algebras except for two pairs. However, these can also be distinguished by deeper methods and so there are 9 algebras arising from the 24 permutations.

Appendix

Part I

Proposition 36. *Let $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H})$, $\mathcal{B} \subseteq \mathcal{B}(\mathcal{K})$ and $\alpha : \mathcal{A} \to \mathcal{B}$ an isometric isomorphism. Then $\alpha(\mathcal{A}) \cap \alpha(\mathcal{A})^* = \alpha(\mathcal{A} \cap \mathcal{A}^*)$; thus $\mathcal{B} \cap \mathcal{B}^* = \alpha(\mathcal{A} \cap \mathcal{A}^*)$.*

Proof. The space $\mathcal{A} \cap \mathcal{A}^*$ is the closed linear space generated by its unitary elements x. Then $\alpha(x) = g = up$, where g is a contraction, u unitary and $0 \leq p \leq I$ invertible. If $\alpha(x^*) = s$, then $sg = \alpha(x^*)\alpha(x) = \alpha(x^*x) = \alpha(I) = I$ and similarly $gs = I$. Thus, $s = p^{-1}u^*(= g^{-1})$. Therefore, $\|s\| \leq 1$, which means that $\|p^{-1}\| \leq 1$ and so $p = I$. Thus, $\alpha(x) = u \in \mathcal{B}$ is unitary. Moreover, $u^* = g = \alpha(x^*) \in \mathcal{B}$, so $\alpha(x) \in \mathcal{B} \cap \mathcal{B}^*$. Thus, $\alpha(\mathcal{A} \cap \mathcal{A}^*) \subseteq \mathcal{B} \cap \mathcal{B}^*$. Likewise, $\alpha^{-1}(\mathcal{B} \cap \mathcal{B}^*) \subseteq \mathcal{A} \cap \mathcal{A}^*$. $\square$

Part II

If $\mathcal{A}_{n,k} \simeq \mathcal{A}_{m,l}$ are isometrically isomorphic, then $\mathcal{E}_k = \mathcal{A}_{n,k} \cap \mathcal{A}^*_{n,k} \simeq \mathcal{A}_{m,l} \cap \mathcal{A}^*_{m,l} = \mathcal{E}_l$ (Cuntz-Toeplitz algebras). In this case $k = l$.

Now, since $\mathcal{A}_n \subseteq \mathcal{A}_{n,k}$, then $\mathfrak{M}(\mathcal{A}_{n,k}) \hookrightarrow \mathfrak{M}(\mathcal{A}_n)$, where

$$\mathfrak{M}(A_{n,k}) \simeq \underbrace{\{0\} \otimes \cdots \otimes \{0\}}_{k-times} \otimes \overline{\mathbb{B}_{n-k}} \hookrightarrow \overline{\mathbb{B}_n} \simeq \mathfrak{M}(\mathcal{A}_n),$$

and

$$\mathfrak{M}(A_{m,k}) \simeq \underbrace{\{0\} \otimes \cdots \otimes \{0\}}_{k-times} \otimes \overline{\mathbb{B}_{m-k}} \hookrightarrow \overline{\mathbb{B}_m} \simeq \mathfrak{M}(\mathcal{A}_m).$$

Since $\mathfrak{M}(\mathcal{A}_{n,k}) \simeq \mathfrak{M}(\mathcal{A}_{m,k})$, we have that $\overline{\mathbb{B}_{n-k}} \simeq \overline{\mathbb{B}_{m-k}}$, so $n - k = m - l$. Thus, $n = m$. So we get the following theorem.

Theorem 37. *$\mathcal{A}_{n,k} \simeq \mathcal{A}_{m,l}$ isometrically isomorphically, iff $n = m$ and $k = l$.*